Math 32B

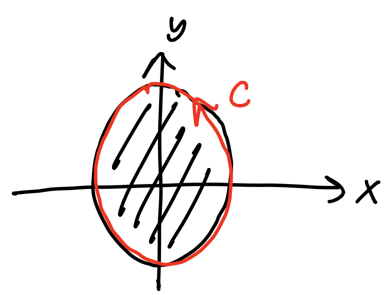
you can ask questions through chat down below or over the microphone

Q #1 Use Green's Thm to find the area of the ellipse $\left(\frac{x}{3}\right)^2+\left(\frac{y}{4}\right)^2=R^2$, where $R>0$.

$$\left(\frac{x}{3R}\right)^2+\left(\frac{y}{4R}\right)^2=1$$

Green's Thm: $$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dA$$

Area $= \iint_D 1 \, dA$



$$1=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$$

$$\vec{F}=\langle P, Q\rangle$$

one ex: $\langle P, Q\rangle = \langle -\frac{1}{2}y, \frac{1}{2}x\rangle$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} - \left(\frac{-1}{2}\right) = 1$$

$$= \int_C \left\langle \frac{-1}{2}y, \frac{1}{2}x \right\rangle \cdot d\vec{r}$$

$\vec{r} = \langle x, y \rangle$ $0 \le t \le 2\pi$

$\vec{r} = \langle 3R\cos t, \underset{\sim}{4R\sin t} \rangle$

$\vec{r}\,'(t) = \langle -3R\sin t, 4R\cos t \rangle$

$$= \int_0^{2\pi} \left\langle \frac{-1}{2}\cdot 4R\sin t, \frac{1}{2}\cdot 3R\cos t \right\rangle \cdot \langle -3R\sin t, 4R\cos t \rangle \, dt$$

$$= \int_0^{2\pi} 6R^2\sin^2 t + 6R^2\cos^2 t \, dt = R^2 6 \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= R^2 6 \int_0^{2\pi} 1 \, dt = R^2 6(2\pi) = 12\pi R^2$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \Rightarrow \pi ab$$

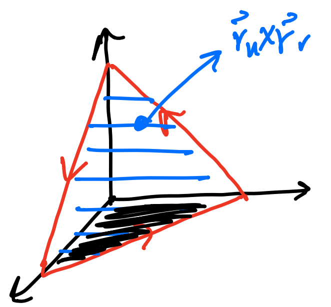
Q: Explain Stokes Theorem w/ a general example

Stokes Theorem $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$

one example: bunch of line integrals we want to avoid, normal is "nice" (planes)

Evaluate $\int_C \vec{F}\cdot d\vec{r}$, $\vec{F} = \langle x+y^2, y+z^2, z+x^2 \rangle$

C is the boundary of $x+y+z=1$ in the first octant



$\text{curl}\,\vec{F} = \langle -2z, -2x, -2y \rangle$

$\vec{r}(x,y,z) = \langle x, y, z \rangle$

$\vec{r}(x,y) = \langle x, y, \underline{1-x-y} \rangle$

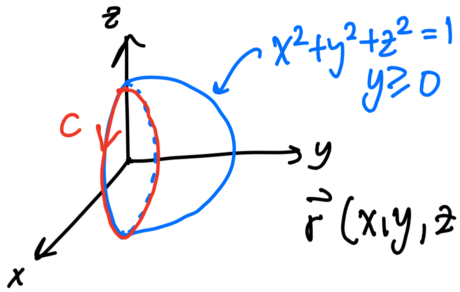
$\vec{r}_x \times \vec{r}_y = \langle 1, 1, 1 \rangle$

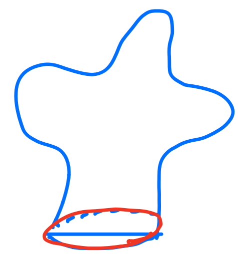
$$\int_0^1\int_0^{1-x} \langle -2(1-x-y), -2x, -2y \rangle \cdot \langle 1, 1, 1 \rangle \, dy\, dx$$

other example: surface normal is tedious
boundary is nice

Evaluate $\iint \operatorname{curl}\vec{F} \cdot d\vec{S}$, $\vec{F} = \langle y, z, x \rangle$,
S is $x^2+y^2+z^2=1$, $y \geq 0$

$$\int_C \vec{F} \cdot d\vec{r}$$





$$\vec{r}(x, y, z) = \langle x, y, z \rangle$$
$$= \langle x, 0, z \rangle$$
$$\vec{r}(r, \theta) = \langle r\cos\theta, 0, r\sin\theta \rangle$$

$$\int \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\, d\theta$$

$$\vec{r}(\theta) = \langle \underbrace{\cos\theta}_{x}, \underbrace{0}_{y}, \underbrace{\sin\theta}_{z} \rangle, \quad 0 \leq \theta \leq 2\pi$$

$$\int_0^{2\pi} \langle 0, \sin\theta, \cos\theta \rangle \cdot \langle -\sin\theta, 0, \cos\theta \rangle \, d\theta$$

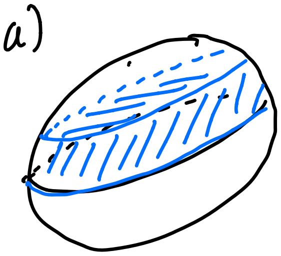
$$\int_0^{2\pi} \cos^2\theta \, d\theta = \cdots$$

#2(c) "Use Stokes' Theorem to compute $\iint \underset{\operatorname{curl}(\quad)}{\vec{F}} \cdot d\vec{S}$"

$$\parallel$$

$$\int_C (\quad) \cdot d\vec{r}$$

#2 (a)



(b) $\text{curl}\langle -y, yz, xz\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & yz & xz \end{vmatrix}$

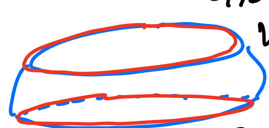
$= \langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz), -\left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(-y)\right),$

$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(-y) \rangle$

$= \langle 0 - y, -(z-0), 0+1 \rangle$

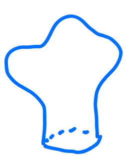
$= \langle -y, -z, 1 \rangle$

(c) $\iint \text{curl}\langle -y, yz, xz\rangle \cdot d\vec{S} = \int_C \langle -y, yz, xz\rangle \cdot d\vec{r}$



$= \int_{C_1} \underbrace{\langle -y, yz, xz\rangle}_{\vec{F}} \cdot d\vec{r}$

$+ \int_{C_2} \langle -y, yz, xz\rangle \cdot d\vec{r}$



C_1: $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{5}\right)^2 = 1$ plug in $z=1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{1}{5}\right)^2 = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \frac{24}{25}$

$$\frac{x^2 \cdot 25}{9^2 \cdot 24} + \frac{y^2\, 25}{16 \cdot 24} = 1$$

Surface

$\downarrow$

line

$$\frac{x^2}{\frac{9^2 \cdot 24}{25}} + \frac{y^2}{\frac{16 \cdot 24}{25}} = 1$$

$$\vec{r}(x, y, z) = \langle x, y, z \rangle$$

$$\vec{r}(x, y\ ) = \langle x,\ y,\ \ 1 \rangle$$

$0 \le \theta \le 2\pi$ $\quad \vec{r}(\theta) = \langle \underbrace{\frac{3\sqrt{24}}{5}\cos\theta}_{x},\ \underbrace{\frac{4\sqrt{24}}{5}\sin\theta}_{y},\ \underbrace{1}_{z} \rangle$

C_2: $\vec{r}(x, y, z) = \langle x, y,\ 0 \rangle$ $\qquad \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

$$\vec{r}(\theta) = \langle \underbrace{3\cos\theta}_{x}, \underbrace{4\sin\theta}_{y}, \underbrace{0}_{z} \rangle \qquad 0 \le \theta \le 2\pi$$

$$\int \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\, d\theta$$

$$\Rightarrow \int_0^{2\pi} \langle -\frac{4\sqrt{24}}{5}\sin\theta,\ \frac{4\sqrt{24}}{5}\sin\theta,\ \frac{3\sqrt{24}}{5}\cos\theta \rangle \cdot \langle -\frac{3\sqrt{24}}{5}\sin\theta,\ \frac{4\sqrt{24}}{5}\cos\theta,\ 0 \rangle\, d\theta$$

$$+ \int_0^{2\pi} \langle -4\sin\theta,\ 4\sin\theta \cdot 0,\ 3\cos\theta \cdot 0 \rangle \cdot \langle -3\sin\theta,\ 4\cos\theta,\ 0 \rangle\, d\theta$$

$= \cdots$

$\vec{A}$ is the vector potential of $\vec{B}$ if curl $\vec{A} = \vec{B}$

#4b contradiction?

$$\underbrace{\int_C \langle x, y, z \rangle \cdot d\vec{r}}_{\swarrow \; \_\_\_\_} = \underbrace{\iint \text{curl}\,\vec{A} \cdot d\vec{r}}_{\text{where curl}\,\vec{A} = \langle x, y, z \rangle} \longrightarrow \_\_\_\_\_\_$$