Math 33A

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If A is a symmetric matrix ($A^T = A$) then there exists Q (orthogonal matrix) and D (diagonal matrix) such that $A = QDQ^T$.

#1a from Week 10 worksheet $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

eigenvalues $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 2\cdot 2$$

$$= \cancel{4} - \lambda - 4\lambda + \lambda^2 - \cancel{4}$$

$$0 = \lambda^2 - 5\lambda$$

$$0 = \lambda(\lambda - 5)$$

$$\lambda_1 = 0 \quad \text{or} \quad \lambda_2 = 5$$

eigenvectors $(A-\lambda I)\vec{v}=\vec{0}$ $\left(\begin{array}{cc|c} 1-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \end{array}\right)$

$\to \lambda_1=0$: $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $x_1+2x_2=0$, x_2 free

$$\vec{v}_1=\begin{pmatrix}-2\\1\end{pmatrix}$$

$\to \lambda_2=5$: $\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $2x_1-x_2=0$, x_1 free

$$\vec{v}_2=\begin{pmatrix}1\\2\end{pmatrix}$$

Gram-Schmidt $\vec{v}_1=\begin{pmatrix}-2\\1\end{pmatrix}$, $\vec{v}_2=\begin{pmatrix}1\\2\end{pmatrix}$

$$\vec{w}_1=\vec{v}_1=\begin{pmatrix}-2\\1\end{pmatrix}$$

$$\vec{w}_2=\vec{v}_2-\text{proj}_{\vec{w}_1}\vec{v}_2=\begin{pmatrix}1\\2\end{pmatrix}-\frac{\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}}{\begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}}\begin{pmatrix}-2\\1\end{pmatrix}$$

$$=\begin{pmatrix}1\\2\end{pmatrix}-\frac{0}{\text{stuff}}\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\vec{u}_1=\frac{\vec{w}_1}{\|\vec{w}_1\|} \qquad \vec{u}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$=\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix} \qquad \vec{u}_2=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$$

$$Q=\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix} \qquad D=\begin{pmatrix}0&0\\0&5\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix} = \underbrace{\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix}}_{Q} \underbrace{\begin{pmatrix}0 & 0\\ 0 & 5\end{pmatrix}}_{D} \underbrace{\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix}}_{Q^T}$$

1. eigenvalues
2. eigenvectors
3. Gram-schmidt

#1C $\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}$ ← doesn't have distinct eigenvalues, so need to do Gram-Schmidt

Q: If 2 3X3 matrices A, B have eigenvalues 1, 2, 3 is A similar to B? yes

$$A = P\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}P^{-1} \leftarrow \begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix} = P^{-1}AP$$

$$B = Q\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}Q^{-1}$$

$$B = QP^{-1}APQ^{-1}$$

$$= \underbrace{QP^{-1}}_{S} A \, (\underbrace{QP^{-1}}_{S})^{-1}$$

similar means

$$B = SAS^{-1}$$

$S = QP^{-1}$ $\quad S^{-1} = (Q^{-1}P)^{-1} = P^{-1}Q$

Ex of not similar matrices $\quad A=\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad B=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

$\lambda=1$, 2 eigenvectors

$\lambda=1$, one eigenvector

$\boxed{A\vec{v}=\lambda\vec{v}} \longrightarrow A\vec{v}-\lambda\vec{v}=\vec{0}$

eigenvector $\quad$ eigenvalue

$(A-\lambda I)\vec{v}=\vec{0} \leftarrow$ eigenvectors

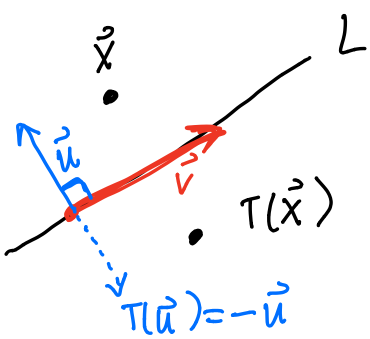
$\det(A-\lambda I)=0 \leftarrow$ eigenvalues

Q: geometric interpretation of eigenvectors

Ex $\quad T:\mathbb{R}^2\to\mathbb{R}^2$

T: reflection about a line L, $\vec{v}$ is a vector on that line

what are the eigenvectors?



one eigenvector is $\vec{v}$

because $T(\vec{v})=\vec{v}$ $\quad(A\vec{v}=\lambda\vec{v})$

eigenvalue 1

another eigenvector is $\vec{u}$

because $T(\vec{u})=-\vec{u}$

eigenvalue -1

#1c worksheet $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ eigenvalues $\lambda_1 = 0, \lambda_2 = 3$

$\lambda_1 = 0$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \lambda_2 = 3 \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \cancel{\vec{w}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}(\vec{v}_3) - \text{proj}_{\vec{w}_2}(\vec{v}_3)$$

SVD

$$A = U \Sigma V^T$$

U: orthogonal matrix

Σ: diagonal matrix

V^T: orthogonal matrix

Ex $A = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}$ ① singular values $= \sqrt{\text{eigenvalues of } A^TA}$

$$A^TA = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}^T \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 85 & -30 \\ -30 & 40 \end{pmatrix} \leftarrow \text{symmetric}$$

$(A^TA)^T = A^TA$

$$\begin{pmatrix} 85 & -30 \\ -30 & 40 \end{pmatrix}^T = \begin{pmatrix} 85 & -30 \\ -30 & 40 \end{pmatrix}$$

$\det(A^TA - \lambda I) = 0$

$$\begin{vmatrix} 85-\lambda & -30 \\ -30 & 40-\lambda \end{vmatrix} = (85-\lambda)(40-\lambda) - (-30)(-30)$$

$$= (\lambda - 100)(\lambda - 25)$$

$$\lambda_1 = 100, \quad \lambda_2 = 25$$

$$\sigma_1 = \sqrt{100}, \quad \sigma_2 = \sqrt{25}$$

$$\sigma_1 = 10, \quad \sigma_2 = 5$$

decreasing order

$$\Sigma = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \qquad \cancel{\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}}$$

② eignvectors of A^TA $\left(\begin{array}{cc|c} 85-\lambda & -30 & 0 \\ -30 & 40-\lambda & 0 \end{array}\right)$

$\lambda_1 = 100$ $\left(\begin{array}{cc|c} -15 & -30 & 0 \\ -30 & -60 & 0 \end{array}\right)$ $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ Gram-schmdt

$\lambda_2 = 25$ $\left(\begin{array}{cc|c} 60 & -30 & 0 \\ -30 & 15 & 0 \end{array}\right)$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\underline{\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}} \qquad \vec{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$V = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$ ← matrix of orthonormal eigenvectors

③ Find U

$A = U \Sigma V^T \longrightarrow AV = U\Sigma$

✓ ✓

$AV\Sigma^{-1} = U$

$(U^TA = \Sigma V^T)$

$U = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$

$$\begin{cases} \vec{u}_1 = \dfrac{1}{\sigma_1} A\vec{v}_1 = \dfrac{1}{10}\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix} = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ \vec{u}_2 = \dfrac{1}{\sigma_2} A\vec{v}_2 = \dfrac{1}{5}\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{cases}$$

$$A = U \Sigma V^T = \boxed{\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T}$$