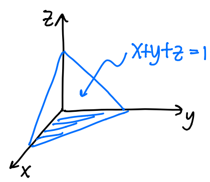
Stokes' Theorem and Divergence Theorem Examples

Written by Victoria Kala, vtkala@math.ucla.edu

Stokes' Theorem: Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

Ex: Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle x+y^2, y+z^2, z+x^2\rangle$, C is the boundary of the part of the plane in the first octant.

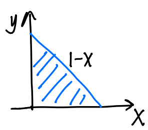


$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_S \operatorname{curl}\vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} = \hat{\imath}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ y+z^2 & z+x^2\end{vmatrix} - \hat{\jmath}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ x+y^2 & z+x^2\end{vmatrix} + \hat{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ x+y^2 & y+z^2\end{vmatrix}$$

$$= \langle 0-2z, -(2x-0), 0-2y\rangle = \langle -2z, -2x, -2y\rangle$$

$$\vec{r}(x,y,z) = \langle x,y,z\rangle$$
$$\vec{r}(x,y) = \langle x, y, 1-x-y\rangle$$



$0\le x\le 1$

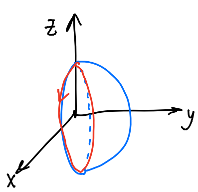
$0\le y\le 1-x$

$$\vec{r}_x = \langle 1,0,-1\rangle$$
$$\vec{r}_y = \langle 0,1,-1\rangle$$

$$\vec{r}_x\times\vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1\end{vmatrix} = \langle 1,1,1\rangle$$

$$\Rightarrow \int_0^1\int_0^{1-x} \langle -2(1-x-y), -2x, -2y\rangle\cdot\langle 1,1,1\rangle\,dy\,dx = \int_0^1\int_0^{1-x}(-2+2x+2y-2x-2y)\,dy\,dx = \int_0^1\int_0^{1-x} -2\,dy\,dx = -2\cdot\tfrac{1}{2}\cdot 1\cdot 1$$
$$= -1$$

Ex: Evaluate $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle y,z,x\rangle$, S is the hemisphere $x^2+y^2+z^2=1$, $y\ge 0$ oriented in the direction of the positive y-axis.



$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

$$\vec{r}(x,y,z) = \langle x,y,z\rangle$$
$$= \langle x,0,z\rangle$$

$$\vec{r}(r,\theta) = \langle r\cos\theta, 0, r\sin\theta\rangle$$
$$\vec{r}(\theta) = \langle \cos\theta, 0, \sin\theta\rangle \qquad 0\le\theta\le 2\pi$$
$$\vec{r}'(\theta) = \langle -\sin\theta, 0, \cos\theta\rangle$$

$$\int_C \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_0^{2\pi}\langle 0,\sin\theta,\cos\theta\rangle\cdot\langle -\sin\theta, 0,\cos\theta\rangle\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = \pi$$

Divergence Theorem: Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV.$$

Ex: Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle x,y,z\rangle$ and S is the surface $x^2+y^2+z^2=1$ w/ outward orientation.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint \operatorname{div}\vec{F}\,dV \qquad \operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$= \iiint 3\,dV = 3\iiint dV = 3\cdot\frac{4}{3}\pi(1)^3 = 4\pi$$