Week 8 Solutions

Math 32B

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1. (a) $\vec{r}(x,y) = \langle x, y, 1+2x+3y \rangle \quad 0 \le x \le 3,\ 0 \le y \le 2$

(b) $\vec{r}(x,y) = \langle x, y, x^2+4y^2 \rangle$

(c) $\vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle \quad 0 \le z \le 1,\ 0 \le \theta \le 2\pi$

(d) $\vec{r}(\theta,\varphi) = \langle a\cos\theta\sin\varphi, a\sin\theta\sin\varphi, a\cos\varphi \rangle \quad 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$

(e) $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle \quad 1 \le r \le 2,\ 0 \le \theta \le 2\pi$ *Note: answers will vary

2. $\vec{r}(0,\pi) = \langle 0\pi, 0\sin\pi, \pi\cos 0 \rangle = \langle 0,0,\pi \rangle \leftarrow (x_0,y_0,z_0) = (0,0,\pi)$

$\vec{r}_u = \langle v, \sin v, -v\sin u \rangle$

$\vec{r}_v = \langle u, u\cos v, \cos u \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v & \sin v & -v\sin u \\ u & u\cos v & \cos u \end{vmatrix} = \hat{\imath}\begin{vmatrix} \sin v & -v\sin u \\ u\cos v & \cos u \end{vmatrix} - \hat{\jmath}\begin{vmatrix} v & -v\sin u \\ u & \cos u \end{vmatrix} + \hat{k}\begin{vmatrix} v & \sin v \\ u & u\cos v \end{vmatrix}$$

$$= \hat{\imath}(\sin v\cos u + uv\sin u\cos v) - \hat{\jmath}(v\cos u + uv\sin u) + \hat{k}(uv\cos v - u\sin v)$$

at $(0,\pi)$, $\vec{r}_u \times \vec{r}_v = \langle \sin\pi\cos 0 + 0\pi\sin 0\cos\pi, -(\pi\cos 0 + 0\pi\sin 0), 0\pi\cos\pi - 0\sin\pi \rangle$

$= \langle 0, -\pi, 0 \rangle$

The plane is $0(x-0) - \pi(y-0) + 0(z-\pi) = 0$

$-\pi y = 0$

$y = 0$

3. (a) $\vec{r}(\theta,\varphi) = \langle a\cos\theta\sin\varphi, a\sin\theta\sin\varphi, a\cos\varphi \rangle \quad 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$

$\vec{r}_\theta = \langle -a\sin\theta\sin\varphi, a\cos\theta\sin\varphi, 0 \rangle$

$\vec{r}_\varphi = \langle a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, -a\sin\varphi \rangle$

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a\sin\theta\sin\varphi & a\cos\theta\sin\varphi & 0 \\ a\cos\theta\cos\varphi & a\sin\theta\cos\varphi & -a\sin\varphi \end{vmatrix} = \hat{\imath}\begin{vmatrix} a\cos\theta\sin\varphi & 0 \\ a\sin\theta\cos\varphi & -a\sin\varphi \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -a\sin\theta\sin\varphi & 0 \\ a\cos\theta\cos\varphi & -a\sin\varphi \end{vmatrix}$$

$$+ \hat{k}\begin{vmatrix} -a\sin\theta\sin\varphi & a\cos\theta\sin\varphi \\ a\cos\theta\cos\varphi & a\sin\theta\cos\varphi \end{vmatrix}$$

$= \langle -a^2\sin^2\varphi\cos\theta, -a^2\sin\theta\sin^2\varphi, -a^2\sin^2\theta\sin\varphi\cos\varphi - a^2\cos^2\theta\sin\varphi\cos\varphi \rangle$

$= \langle -a^2\sin^2\varphi\cos\theta, -a^2\sin\theta\sin^2\varphi, -a^2\sin\varphi\cos\varphi \rangle$ since $\sin^2\theta + \cos^2\theta = 1$

$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \sqrt{a^4\sin^4\varphi\cos^2\theta + a^4\sin^2\theta\sin^4\varphi + a^4\sin^2\varphi\cos^2\varphi}$

$= \sqrt{a^4\sin^4\varphi + a^4\sin^2\varphi\cos^2\varphi}$ since $\sin^2\theta + \cos^2\theta = 1$

$= \sqrt{a^4\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)}$

$= a^2\sin\varphi$ since $\sin^2\varphi + \cos^2\varphi = 1$

$$A = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi\, d\varphi\, d\theta = a^2\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi} d\varphi\, d\theta = a^2\int_0^{2\pi}(-\cos\pi + \cos 0)\, d\theta = 2a^2\int_0^{2\pi} d\theta = 4\pi a^2$$

③ continued

(b) From 1c, $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$ $\quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$\vec{r}_r = \langle \cos\theta, \sin\theta, 1 \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\cos^2\theta + r\sin^2\theta \rangle$$

$= \langle -r\cos\theta, -r\sin\theta, r \rangle$ since $\cos^2\theta + \sin^2\theta = 1$

$$\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$$

$$A = \int_0^{2\pi}\int_0^1 \sqrt{2}r \, dr\, d\theta = \sqrt{2}\int_0^{2\pi} \frac{r^2}{2}\Big|_0^1 d\theta = \frac{\sqrt{2}}{2}\int_0^{2\pi} d\theta = \frac{\sqrt{2}}{2}\cdot 2\pi = \sqrt{2}\pi$$

④(a) From 1a, $\vec{r}(x,y) = \langle x, y, 1+2x+3y \rangle$ $\quad 0 \le x \le 3, \ 0 \le y \le 2$

$\vec{r}_x = \langle 1, 0, 2 \rangle$

$\vec{r}_y = \langle 0, 1, 3 \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = \langle -2, -3, 1 \rangle \qquad \|\vec{r}_x \times \vec{r}_y\| = \sqrt{4+9+1} = \sqrt{14}$$

$$\iint_S x^2yz\, dS = \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{14}\, dy\, dx = \sqrt{14}\int_0^3\int_0^2 (x^2y + 2x^3y + 3x^2y^2)\, dy\, dx$$

$$= \sqrt{14}\int_0^3 \left(\frac{x^2y^2}{2} + x^3y^2 + x^2y^3 \right)\Big|_{y=0}^2 dx = \sqrt{14}\int_0^3 (2x^2 + 4x^3 + 8x^2)\, dx = \sqrt{14}\int_0^3 (4x^3 + 10x^2)\, dx$$

$$= \sqrt{14}\left(x^4 + \frac{10}{3}x^3\right)\Big|_{x=0}^3 = \sqrt{14}\left(81 + \frac{10}{3}\cdot 27\right) = \sqrt{14}(81 + 90) = \boxed{171\sqrt{14}}$$

④(b) From 1c, $\vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$ $\quad 0 \le z \le 1, \ 0 \le \theta \le 2\pi$

$\vec{r}_\theta = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$

$\vec{r}_z = \langle 0, 0, 1 \rangle$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 3\cos\theta, 3\sin\theta, 0 \rangle$$

$$\|\vec{r}_\theta \times \vec{r}_z\| = \sqrt{9\cos^2\theta + 9\sin^2\theta + 0^2} = \sqrt{9} = 3$$

$$\iint y\, dS = \int_0^{2\pi}\int_0^1 3\sin\theta \cdot 3\, dz\, d\theta = 9\int_0^{2\pi} z\Big|_{z=0}^1 \sin\theta\, d\theta = 9(-\cos\theta)\Big|_0^{2\pi} = 0$$