Week 10 Solutions

Math 33A
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① (a) $\det(A-\lambda I)=0$: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)-4 = 4-\lambda-4\lambda+\lambda^2-4 = \lambda^2-5\lambda = \lambda(\lambda-5)=0$ $\boxed{\lambda_1=0,\ \lambda_2=5}$

$(A-\lambda I)\vec{v}=\vec{0}$: $\left(\begin{array}{cc|c} 1-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \end{array}\right)$

$\lambda_1=0$: $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $x_1=-2x_2$, x_2 free $\boxed{\vec{v}_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

$\lambda_2=5$: $\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $2x_1=x_2$, x_1 free $\boxed{\vec{v}_2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

(b) $\det(B-\lambda I)=0$: $\begin{vmatrix} 6-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda)-4 = 18-6\lambda-3\lambda+\lambda^2-4 = \lambda^2-9\lambda+14 = (\lambda-7)(\lambda-2)=0$ $\boxed{\lambda_1=7,\ \lambda_2=2}$

$(B-\lambda I)\vec{v}=\vec{0}$: $\left(\begin{array}{cc|c} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \end{array}\right)$

$\lambda_1=7$: $\left(\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $x_1=-2x_2$, x_2 free $\boxed{\vec{v}_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

$\lambda_2=2$: $\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $2x_1=x_2$, x_1 free $\boxed{\vec{v}_2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

(c) $\det(C-\lambda I)=0$: $\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} -1 & 2-\lambda \\ -1 & -1 \end{vmatrix}$

$= (2-\lambda)\left((2-\lambda)^2-1\right) + 1\left(-2+\lambda-1\right) - 1\left(1+2-\lambda\right)$

$= (2-\lambda)\left(4-4\lambda+\lambda^2-1\right) + \lambda-3 + \lambda-3$

$= (2-\lambda)\left(\lambda^2-4\lambda+3\right) + 2\lambda-6$

$= 2\lambda^2-8\lambda+\cancel{6}-\lambda^3+4\lambda^2-3\lambda+2\lambda-\cancel{6}$

$= -\lambda^3+6\lambda^2-9\lambda$

$= -\lambda(\lambda^2-6\lambda+9)$

$0 = -\lambda(\lambda-3)^2$

$\boxed{\lambda_1=0,\ \lambda_2=3}$

$(C-\lambda I)\vec{v}=\vec{0}$: $\left(\begin{array}{ccc|c} 2-\lambda & -1 & -1 & 0 \\ -1 & 2-\lambda & -1 & 0 \\ -1 & -1 & 2-\lambda & 0 \end{array}\right)$

$\lambda_1=0$: $\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right)$ $R_1 \leftrightarrow R_2$ $\left(\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right)$ $\begin{matrix} 2R_1+R_2 \to R_2 \\ R_3-R_1 \to R_1 \end{matrix}$ $\left(\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$ $R_3+R_2 \to R_3$ $\left(\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{matrix} -R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \end{matrix}$

$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $2R_2+R_1 \to R_1$ $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1=x_3$, $x_2=x_3$, x_3 free $\boxed{\vec{v}_1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

$\lambda_2=3$: $\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right)$ $\begin{matrix} R_2-R_1 \to R_2 \\ R_3-R_1 \to R_3 \end{matrix}$ $\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1=-x_2-x_3$, x_2, x_3 free

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}x_3$

$\boxed{\vec{v}_2=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ \vec{v}_3=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

② (a) $\vec{v}_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\vec{v}_2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\vec{w}_2 = \vec{v}_2 - \mathrm{proj}_{\vec{w}_1}(\vec{v}_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \dfrac{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -2 \\ 1 \end{pmatrix}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \vec{0} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$Q = \begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

Check: $$\boxed{QDQ^T = \begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix}} = \begin{pmatrix} 0 & \sqrt5 \\ 0 & 2\sqrt5 \end{pmatrix}\begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = A \checkmark$$

(b) $\vec v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\vec v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ same vectors as (a) $\longrightarrow$ $\vec u_1 = \frac{1}{\sqrt5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\vec u_2 = \frac{1}{\sqrt5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$Q = \begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix} \qquad D = \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}$$

Check: $$\boxed{QDQ^T = \begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix}\begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix}} = \begin{pmatrix} -14/\sqrt5 & 2/\sqrt5 \\ 7/\sqrt5 & 4/\sqrt5 \end{pmatrix}\begin{pmatrix} -2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{pmatrix} = \begin{pmatrix} \frac{28}{5}+\frac{2}{5} & \frac{-14}{5}+\frac{4}{5} \\ \frac{-14}{5}+\frac{4}{5} & \frac{7}{5}+\frac{8}{5} \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$$
$= B \checkmark$

(c) $\vec v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\vec w_1 = \vec v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec w_2 = \vec v_2 - \mathrm{proj}_{\vec w_1}(\vec v_2) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \vec 0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec w_3 = \vec v_3 - \mathrm{proj}_{\vec w_1}(\vec v_3) - \mathrm{proj}_{\vec w_2}(\vec v_3) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \vec 0 - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\vec u_1 = \frac{1}{\sqrt3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec u_2 = \frac{1}{\sqrt2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec u_3 = \frac{1}{\sqrt6}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1/\sqrt3 & -1/\sqrt2 & -1/\sqrt6 \\ 1/\sqrt3 & 1/\sqrt2 & -1/\sqrt6 \\ 1/\sqrt3 & 0 & 2/\sqrt6 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Check: $$\boxed{QDQ^T = \begin{pmatrix} 1/\sqrt3 & -1/\sqrt2 & -1/\sqrt6 \\ 1/\sqrt3 & 1/\sqrt2 & -1/\sqrt6 \\ 1/\sqrt3 & 0 & 2/\sqrt6 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1/\sqrt3 & 1/\sqrt3 & 1/\sqrt3 \\ -1/\sqrt2 & 1/\sqrt2 & 0 \\ -1/\sqrt6 & -1/\sqrt6 & 2/\sqrt6 \end{pmatrix}} = \begin{pmatrix} 0 & -3/\sqrt2 & -3/\sqrt6 \\ 0 & 3/\sqrt2 & -3/\sqrt6 \\ 0 & 0 & 6/\sqrt6 \end{pmatrix}\begin{pmatrix} 1/\sqrt3 & 1/\sqrt3 & 1/\sqrt3 \\ -1/\sqrt2 & 1/\sqrt2 & 0 \\ -1/\sqrt6 & -1/\sqrt6 & 2/\sqrt6 \end{pmatrix}$$

$$= \begin{pmatrix} 0+\frac{3}{2}+\frac{3}{6} & 0-\frac{3}{2}+\frac{3}{6} & 0+0-\frac{6}{6} \\ 0-\frac{3}{2}+\frac{3}{6} & 0+\frac{3}{2}+\frac{3}{6} & 0+0-\frac{6}{6} \\ 0+0-\frac{6}{6} & 0+0-\frac{6}{6} & 0+0+\frac{12}{6} \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = C \checkmark$$