Week 8 Solutions

Math 33A
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1. $A\vec{x}=\vec{b} \Rightarrow A^TA\vec{x}^*=A^T\vec{b}$ $\quad \begin{pmatrix}1&-1&1\\1&1&2\end{pmatrix}\begin{pmatrix}1&1\\-1&1\\1&2\end{pmatrix}\vec{x}^*=\begin{pmatrix}1&-1&1\\1&1&2\end{pmatrix}\begin{pmatrix}11\\3\\4\end{pmatrix}$

$$\begin{pmatrix}3&2\\2&6\end{pmatrix}\vec{x}^*=\begin{pmatrix}12\\22\end{pmatrix}$$

$$x^*=\begin{pmatrix}3&2\\2&6\end{pmatrix}^{-1}\begin{pmatrix}12\\22\end{pmatrix}=\frac{1}{18-4}\begin{pmatrix}6&-2\\-2&3\end{pmatrix}\begin{pmatrix}12\\22\end{pmatrix}=\frac{1}{14}\begin{pmatrix}72-44\\-24+66\end{pmatrix}=\frac{1}{14}\begin{pmatrix}28\\42\end{pmatrix}=\boxed{\begin{pmatrix}2\\3\end{pmatrix}}$$

2. $(1,1)\longrightarrow 1=a+b$
$(3,2)\longrightarrow 2=3a+b$
$(4,6)\longrightarrow 6=4a+b$

$$\begin{pmatrix}1&1\\3&1\\4&1\end{pmatrix}\underbrace{\begin{pmatrix}a\\b\end{pmatrix}}_{\vec{x}}=\begin{pmatrix}1\\2\\6\end{pmatrix}\Rightarrow\begin{pmatrix}1&3&4\\1&1&1\end{pmatrix}\begin{pmatrix}1&1\\3&1\\4&1\end{pmatrix}\vec{x}^*=\begin{pmatrix}1&3&4\\1&1&1\end{pmatrix}\begin{pmatrix}1\\2\\6\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}26&8\\8&3\end{pmatrix}\vec{x}^*=\begin{pmatrix}31\\9\end{pmatrix}\Rightarrow\vec{x}^*=\begin{pmatrix}26&8\\8&3\end{pmatrix}^{-1}\begin{pmatrix}31\\9\end{pmatrix}=\frac{1}{78-64}\begin{pmatrix}3&-8\\-8&26\end{pmatrix}\begin{pmatrix}31\\9\end{pmatrix}=\frac{1}{14}\begin{pmatrix}93-72\\-248+234\end{pmatrix}=\begin{pmatrix}21\\-14\end{pmatrix}=\begin{pmatrix}3/2\\-1\end{pmatrix}=\begin{pmatrix}a^*\\b^*\end{pmatrix}$$

$$\boxed{y=\frac{3}{2}x-1}$$

3. (a) $\vec{w}_1=v_1=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$$\vec{w}_2=\vec{v}_2-\text{proj}_{\vec{w}_1}(\vec{v}_2)=\begin{pmatrix}1\\2\\4\\5\end{pmatrix}-\frac{\begin{pmatrix}1\\2\\4\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\2\\4\\5\end{pmatrix}-\frac{12}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}1-3\\2-3\\4-3\\5-3\end{pmatrix}=\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}$$

$$\vec{u}_1=\frac{\vec{w}_1}{\|\vec{w}_1\|}=\frac{1}{\sqrt{4}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\boxed{\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}}\qquad \vec{u}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}=\frac{1}{\sqrt{10}}\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}=\boxed{\begin{pmatrix}-2/\sqrt{10}\\-1/\sqrt{10}\\1/\sqrt{10}\\2/\sqrt{10}\end{pmatrix}}$$

(b) $Q=\begin{pmatrix}1/2&-2/\sqrt{10}\\1/2&-1/\sqrt{10}\\1/2&1/\sqrt{10}\\1/2&2/\sqrt{10}\end{pmatrix}$ $\quad R=\begin{pmatrix}\vec{u}_1\cdot\vec{v}_1&\vec{u}_1\cdot\vec{v}_2\\0&\vec{u}_2\cdot\vec{v}_2\end{pmatrix}=\begin{pmatrix}\frac12+\frac12+\frac12+\frac12&\frac12+\frac22+\frac42+\frac52\\0&\frac{-2}{\sqrt{10}}-\frac{2}{\sqrt{10}}+\frac{4}{\sqrt{10}}+\frac{10}{\sqrt{10}}\end{pmatrix}=\begin{pmatrix}2&6\\0&\sqrt{10}\end{pmatrix}$

$$\boxed{A=\begin{pmatrix}1/2&-2/\sqrt{10}\\1/2&-1/\sqrt{10}\\1/2&1/\sqrt{10}\\1/2&2/\sqrt{10}\end{pmatrix}\begin{pmatrix}2&6\\0&\sqrt{10}\end{pmatrix}}$$

(c) $(1,1,2)\rightarrow a+b-2=0$
$(1,2,1)\rightarrow a+2b-1=0$
$(1,4,1)\rightarrow a+4y-1=0$
$(1,5,2)\rightarrow a+5b-2=0$

$$\underbrace{\begin{pmatrix}1&1\\1&2\\1&4\\1&5\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}a\\b\end{pmatrix}}_{\vec{x}}=\underbrace{\begin{pmatrix}2\\1\\1\\2\end{pmatrix}}_{\vec{b}}\qquad \text{do } A^TA\vec{x}^*=A^T\vec{b}$$

one way:

$$\begin{pmatrix}1&1&1&1\\1&2&4&5\end{pmatrix}\begin{pmatrix}1&1\\1&2\\1&4\\1&5\end{pmatrix}\vec{x}^*=\begin{pmatrix}1&1&1&1\\1&2&4&5\end{pmatrix}\begin{pmatrix}2\\1\\1\\2\end{pmatrix}\Rightarrow\begin{pmatrix}4&12\\12&46\end{pmatrix}\vec{x}^*=\begin{pmatrix}6\\18\end{pmatrix}$$

$$\vec{x}^*=\begin{pmatrix}4&12\\12&46\end{pmatrix}^{-1}\begin{pmatrix}6\\18\end{pmatrix}=\frac{1}{184-169}\begin{pmatrix}46&-13\\-13&4\end{pmatrix}\begin{pmatrix}6\\18\end{pmatrix}=\frac{1}{15}\begin{pmatrix}276-234\\-78+72\end{pmatrix}=\begin{pmatrix}42/15\\-6/15\end{pmatrix}=\begin{pmatrix}14/5\\-2/5\end{pmatrix}$$

③ (c) continued

another way since $A = QR$ from (b), then $A^TA\vec{x}^* = A^T\vec{b}$

$$(QR)^T(QR)\vec{x}^* = A^T\vec{b}$$
$$R^TQ^TQR\vec{x}^* = A^T\vec{b}$$
$$R^TR\vec{x}^* = A^T\vec{b} \quad \text{since } Q^TQ = I$$

$$\Rightarrow \begin{pmatrix} 2 & 0 \\ 6 & \sqrt{10} \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 0 & \sqrt{10} \end{pmatrix}\vec{x}^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 12 \\ 12 & 46 \end{pmatrix}\vec{x}^* = \begin{pmatrix} 6 \\ 18 \end{pmatrix}$$, same system as above

$$\vec{x}^* = \begin{pmatrix} 14/5 \\ -2/5 \end{pmatrix} = \begin{pmatrix} a^* \\ b^* \end{pmatrix}$$

$$a^*x + b^*y - z = 0$$

$$\boxed{\frac{14}{5}x - \frac{2}{5}y - z = 0}$$