Week 9 Solutions

Math 33A
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① (a) $\begin{vmatrix}1&1\\4&-2\end{vmatrix} = 1(-2)-1(4) = -2-4 = -6$

(b) $\begin{vmatrix}0&0&-2\\1&2&1\\1&0&3\end{vmatrix} = 0\begin{vmatrix}2&1\\0&3\end{vmatrix} - 0\begin{vmatrix}1&1\\1&3\end{vmatrix} + (-2)\begin{vmatrix}1&2\\1&0\end{vmatrix} = -2(0-2) = 4$

(c) $\begin{vmatrix}1&0&0\\0&2&0\\0&0&3\end{vmatrix} = 1\begin{vmatrix}2&0\\0&3\end{vmatrix} - 0\begin{vmatrix}0&0\\0&3\end{vmatrix} + 0\begin{vmatrix}0&2\\0&0\end{vmatrix} = 1\cdot 2\cdot 3 = 6$

(d) $\begin{vmatrix}1&0&0\\1&3&0\\1&3&5\end{vmatrix} = 1\begin{vmatrix}3&0\\3&5\end{vmatrix} - 0\begin{vmatrix}1&0\\1&5\end{vmatrix} + 0\begin{vmatrix}1&3\\1&3\end{vmatrix} = 1\cdot 3\cdot 5 = 15$

② yes, all exist because their determinants are nonzero

③ product of diagonal entries

④ product of diagonal entries

⑤ (a) 1

(b) $\det(A)\det(B)$

(c) $\det(A)$

⑥ $AA^{-1} = I$ take det of both sides $\Rightarrow$ $\det(A)\det(A^{-1}) = 1$

$\boxed{\det(A^{-1}) = \dfrac{1}{\det A}}$ or $\boxed{\det(A^{-1}) = (\det A)^{-1}}$

⑦ $\det A = -6$, $\boxed{\det(A^{-1}) = -\frac{1}{6}}$ $\quad$ $\det B = 4$, $\boxed{\det(B^{-1}) = \frac{1}{4}}$ $\quad$ $\det C = 6$, $\boxed{\det(C^{-1}) = \frac{1}{6}}$ $\quad$ $\det D = 15$, $\boxed{\det(D^{-1}) = \frac{1}{15}}$

⑧ $Q^TQ = I$ take det of both sides $\Rightarrow$ $(\det Q)^2 = 1$ $\Rightarrow$ $\boxed{\det Q = \pm 1}$

⑨ A is $n\times n$ matrix $\Rightarrow$

$$\det(A^3) = \det(4A)$$
$$[\det(A)]^3 = 4^n \det A$$
$$(\det A)^3 - 4^n \det A = 0$$
$$\det A\left((\det A)^2 - 4^n\right) = 0$$

So $\underbrace{\det A = 0}_{\uparrow\ A \text{ may not be invertible}}$ or $(\det A)^2 = 4^n$

no

⑩ (a) $\det(A-\lambda I)=0$: $\begin{vmatrix}1-\lambda&1\\4&-2-\lambda\end{vmatrix} = (1-\lambda)(-2-\lambda)-4 = -2-\lambda+2\lambda+\lambda^2-4 = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)=0 \Rightarrow \boxed{\lambda_1=-3,\ \lambda_2=2}$

$(A-\lambda I)\vec{v}=\vec{0}$: $\left(\begin{array}{cc|c}1-\lambda&1&0\\4&-2-\lambda&0\end{array}\right)$

$\lambda_1=-3$: $\left(\begin{array}{cc|c}4&1&0\\4&1&0\end{array}\right) \to \left(\begin{array}{cc|c}4&1&0\\0&0&0\end{array}\right)$ $\quad 4x_1+x_2=0$, x_1 free $\quad \boxed{\vec{v}_1 = \begin{pmatrix}1\\-4\end{pmatrix}}$

$\lambda_2=2$: $\left(\begin{array}{cc|c}-1&1&0\\4&-4&0\end{array}\right) \to \left(\begin{array}{cc|c}-1&1&0\\0&0&0\end{array}\right)$ $\quad -x_1+x_2=0$, x_2 free $\quad \boxed{\vec{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}}$

(b) $\det(E-\lambda I)=0$: $\begin{vmatrix}1-\lambda&-2\\0&3-\lambda\end{vmatrix} = (1-\lambda)(3-\lambda)-0 = (1-\lambda)(3-\lambda)=0$ $\quad \boxed{\lambda_1=1,\ \lambda_2=3}$

$(E-\lambda I)\vec{v}=\vec{0}$: $\left(\begin{array}{cc|c}1-\lambda&-2&0\\0&3-\lambda&0\end{array}\right)$

$\lambda_1=1$: $\left(\begin{array}{cc|c}0&-2&0\\0&2&0\end{array}\right) \to \left(\begin{array}{cc|c}0&1&0\\0&0&0\end{array}\right)$ $\quad x_2=0$, x_1 free $\quad \boxed{\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}}$

$\lambda_2=3$: $\left(\begin{array}{cc|c}-2&-2&0\\0&0&0\end{array}\right) \to \left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right)$ $\quad x_1+x_2=0$, x_2 free $\quad \boxed{\vec{v}_2 = \begin{pmatrix}-1\\1\end{pmatrix}}$

⑪ diagonal

(12) C has eigenvalues $\lambda = 1, 2, 3$

D has eigenvalues $\lambda = 1, 3, 5$

(13) (a) $\det(B-\lambda I) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = -0\begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix} + (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} - 0\begin{vmatrix} -\lambda & -2 \\ 1 & 1 \end{vmatrix} = (2-\lambda)\left[-\lambda(3-\lambda)+2\right] = (2-\lambda)(-3\lambda+\lambda^2+2)$

$= -6\lambda + 2\lambda^2 + 4 + 3\lambda^2 - \lambda^3 - 2\lambda$

$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$ ✓

(b) $\det(B-\lambda I) = 0 \Rightarrow -(\lambda-2)^2(\lambda-1) = 0$ $\boxed{\lambda_1 = 2,\ \lambda_2 = 1}$

(c) $(B-\lambda I)\vec{v} = \vec{0}$: $\left(\begin{array}{ccc|c} -\lambda & 0 & -2 & 0 \\ 1 & 2-\lambda & 1 & 0 \\ 1 & 0 & 3-\lambda & 0 \end{array}\right)$

$\lambda_1 = 2$: $\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = -x_3$, x_2, x_3 free $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s \\ t \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} s$ $\boxed{\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$\lambda_2 = 1$: $\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = -2x_3$, $x_2 = x_3$, x_3 free $\boxed{\vec{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}$

(14) distinct

(15) An nxn matrix is not diagonalizable when it does not have n eigenvectors e.g. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable, it has only one eigenvector

(16) $A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}^{-1}$ $B = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$

$E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1}$

(17) $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$

so $A^{2020} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}^{2020}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} (-3)^{2020} & 0 \\ 0 & 2^{2020} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} (-3)^{2020} & 2^{2020} \\ -4(-3)^{2020} & 2^{2020} \end{pmatrix}\frac{1}{1+4}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$

$= \boxed{\frac{1}{5}\begin{pmatrix} (-3)^{2020} + 4\cdot 2^{2020} & -(-3)^{2020} + 2^{2020} \\ -4(-3)^{2020} + 4\cdot 2^{2020} & 4(-3)^{2020} + 2^{2020} \end{pmatrix}}$

Similarly, $E^{2020} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{2020}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1^{2020} & 0 \\ 0 & 3^{2020} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3^{2020} \\ 0 & 3^{2020} \end{pmatrix}\frac{1}{1-0}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 1 & 1-3^{2020} \\ 0 & 3^{2020} \end{pmatrix}}$

$B^{2020} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{2020}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \boxed{\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{2020} & 0 & 0 \\ 0 & 2^{2020} & 0 \\ 0 & 0 & 1^{2020} \end{pmatrix}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}}$