Math 151A - Spring 2020 - Week 3

Today:

- Fixed-Point Iteration
- Homework questions (if time)

Important Theorems:

- Intermediate Value Theorem: If f is continuous on $[a, b]$ with $f(a)$ and $f(b)$ having opposite signs, then there exists a $c \in (a, b)$ such that $f(c) = 0$.
- Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(a) Show that g has a fixed point on this interval.

Want to show $\exists\ p \in [1,2]$ such that $g(p) = p$

$\Leftrightarrow g(p) - p = 0$

Define $f(x) = g(x) - x$

$f(1) = \arctan 1 + \frac{1}{2} - 1 > 0$

$f(2) = \arctan 2 + \frac{1}{2} - 2 < 0$

f is continuous

By IVT $\exists\ p \in (1,2)$ such that $f(p) = 0$

$\Rightarrow g(p) = p$.

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(b) Show that g has a unique fixed point on this interval.
Option 1: Use derivatives

$f(x) = g(x) - x$, by part (a) we know $\exists\ p \in (1,2)$ such that $f(p) = 0$

$f'(x) = g'(x) - 1 = \frac{1}{1+x^2} - 1 < 0$ strictly decreasing in $(1,2)$

$1 \leq x \leq 2 \Rightarrow \frac{1}{5} \leq \frac{1}{1+x^2} \leq \frac{1}{2}$

$\Downarrow$

p is unique

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(b) Show that g has a unique fixed point on this interval.
Option 2: Contradiction using Mean Value Theorem

Suppose by contradiction $\exists$ p, q, $p \neq q$, s.t. $g(p) = p$ & $g(q) = q$.
By MVT $\exists$ r between p and q such that

$$|g(p) - g(q)| = |g'(r)||(p-q)|$$

$$g'(r) = \frac{1}{1+r^2} \quad \text{since } [1,2] \quad |g'(r)| \leq \frac{1}{2}$$

$$|g(p) - g(q)| \leq \frac{1}{2}|p-q| < |p-q|$$

$$|p-q| < |p-q|$$

Contradiction
$\Rightarrow$ the fixed point is unique

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(c) Suppose we wish to use the fixed-point iteration for approximating p. Does the method converge? Justify your answer.

Option 1: Prove directly with Mean Value Theorem

Fixed-point iteration

$(g(p) = p)$

$p_0 \in [1,2]$

$p_n = g(p_{n-1}), \ n \geq 1$

want to show $p_n \to p$

$\Leftrightarrow |p_n - p| \to 0$ as $n \to \infty$

$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(r_{n-1})||p_{n-1} - p|$ by MVT where r_{n-1} is between p_{n-1} and p

$= \left|\frac{1}{1+r_{n-1}^2}\right| |p_{n-1} - p|$ $\quad r_{n-1} \in [1,2]$

$\leq \frac{1}{2} |p_{n-1} - p|$

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(c) Suppose we wish to use the fixed-point iteration for approximating p. Does the method converge? Justify your answer.

Option 1: Prove directly with Mean Value Theorem

$$|p_n - p| \leq \frac{1}{2}|p_{n-1} - p| = \frac{1}{2}|g(p_{n-2}) - g(p)|$$

$$= \frac{1}{2}|g'(r_{n-2})||p_{n-2} - p|$$

by MVT where r_{n-2} is between p_{n-2} and p, $r_{n-2} \in [1,2]$

$$= \frac{1}{2}\left|\frac{1}{1+r_{n-2}^2}\right||p_{n-2} - p|$$

$$\leq \left(\frac{1}{2}\right)^2 |p_{n-2} - p|$$

$\vdots$

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(c) Suppose we wish to use the fixed-point iteration for approximating p. Does the method converge? Justify your answer.

Option 1: Prove directly with Mean Value Theorem

$$|p_n - p| \le \left(\frac{1}{2}\right)^n |p_0 - p|$$

$$\left(\frac{1}{2}\right)^n \to 0 \quad \text{as } n \to \infty$$

$$|p_n - p| \to 0 \quad \text{as } n \to \infty$$

$p_n \to p$ as $n \to \infty$ iteration converges ✓

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(c) Suppose we wish to use the fixed point iteration for approximating p. Does the method converge? Justify your answer.
Option 2: Use the **Fixed-Point Theorem** (Theorem 2.4 of textbook)

g is continuous on [a, b] ✓

Fixed-Point Theorem. Let $g \in C[a, b]$ such that $g(x) \in [a, b]$ ✓ for all $x \in [a, b]$. Suppose, in addition that g' exists on (a, b) ✓ and that a constant $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$. Then for any number p_0 in $[a, b]$ the sequence defined by $p_n = g(p_{n-1}), n \geq 1$ converges to the unique fixed point p in $[a, b]$.

$$|g'(x)| = \left|\frac{1}{1+x^2}\right| \leq \underbrace{\frac{1}{2}}_{k}$$

Since all hypotheses are satisfied, then by Fixed-point theorem the iteration converges.

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(d) Estimate the number of iterations necessary to achieve an accuracy of 10^{-3} when applying the fixed point iteration for approximating p.
Option 1: Use bound from part (c), Option 1

$$|p_n - p| \leq \left(\frac{1}{2}\right)^n |p_0 - p| \leq \left(\frac{1}{2}\right)^n |b-a| = \underline{\underline{\left(\frac{1}{2}\right)^n}}$$

p_0? p?

$$\left(\frac{1}{2}\right)^n < 10^{-3}$$

$$\frac{1}{2^n} < \frac{1}{10^3} \Rightarrow 2^n > 10^3 \Rightarrow n > \frac{\log(10^3)}{\log 2}$$

$\simeq 9.96$ ith

$\boxed{10 \text{ iterations}}$

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Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(d) Estimate the number of iterations necessary to achieve an accuracy of 10^{-3} when applying the fixed point iteration for approximating p.
Option 2: Use **Corollary 2.5** of the textbook

Corollary 2.5 Let $g \in C[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition that g' exists on (a, b) and that a constant $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$. Then the bounds for the error of fixed-point iteration involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\} \tag{1}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| \tag{2}$$

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Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.
a b

(d) Estimate the number of iterations necessary to achieve an accuracy of 10^{-3} when applying the fixed point iteration for approximating p.
Option 2: Use **Corollary 2.5** of the textbook

$$|p_n - p| \le k^n \max\{p_0 - a, b - p_0\} \le k^n |b-a| \quad (1)$$

?

$k^n |b-a| < 10^{-3}$

$\left(\frac{1}{2}\right)^n < 10^{-3}$ same as last, $n = 10$ iterations

$|g'(x)| \le k$

$\left|\frac{1}{1+x^2}\right| \le \frac{1}{2}$ on $[1,2]$

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(d) Estimate the number of iterations necessary to achieve an accuracy of 10^{-3} when applying the fixed point iteration for approximating p.
Option 2: Use **Corollary 2.5** of the textbook

$k = \frac{1}{2}$

$$|p_n - p| \leq \frac{k^n}{1-k}|\underset{?}{p_1} - \underset{?}{p_0}| \leq \frac{k^n}{1-k}|b-a| \qquad (2)$$

$$\frac{k^n}{1-k} < 10^{-3}$$

$$k^n < 10^{-3}(1-k)$$

$$\left(\frac{1}{2}\right)^n < 10^{-3} \cdot \frac{1}{2}$$

$$\frac{1}{2^n} < \frac{1}{2 \cdot 10^{-3}}$$

$$2^n > 2 \cdot 10^{-3}$$

$$n > \frac{\log(2 \cdot 10^{-3})}{\log(2)} \approx 10.96\text{ish}$$

$\boxed{\text{11 iterations}}$

Example. (Similar to Exercise 4 on Homework 2)
Consider the function $g(x) = \arctan x + \frac{1}{2}$ on the interval $[1, 2]$.

(e) Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-3} using stopping criteria based on $|g(p_n) - p_n|$ and $|p_n - p_{n-1}|$. Create three figures for the following convergence histories: $|g(p_n) - p_n|, |p_n - p_{n-1}|$, and $|p_n - p^*|$ where p^* is your approximation for p.

Homework 1 Announcements:

- If your number of iterations for bisection method doesn't match your theoretical calculation, make the following change in `bisection.m`:

$$\texttt{if err < tol } \underline{\texttt{|| abs(fp) < tol}} \quad \rightarrow \quad \texttt{if err < tol}$$

- For Exercise 3, you know that $\underline{p = 0.7}$. Use this value for plotting $|p - p_n|$ vs. iteration.
- For Exercise 4, you do not know the exact value of p. You do not need to plot $|p - p_n|$ vs. iteration.

$|\tilde{p} - p_n|$ where $\tilde{p}$ is approximate p

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Questions?