Math 151A - Spring 2020 - Week 4

Today:

- Order of convergence
- Comparing iterative methods

Homework Announcements:

- Make sure to "submit" your homework on CCLE, not just upload a draft
- Cite any important theorems you use and why you can use them
- For programming questions, make sure to write a short summary of your result (e.g. "My code estimated p = 1.365230 with a tolerance 10^{-6} and initial guess $p_0 = 1.5$")
- Export images as .png or `.jpg`, don't use the Matlab default `.fig`
- If you submitted a code but it has errors, make sure to fix before the exam

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If p_n is a sequence converging to p with $p_n \neq p$ for all n, and if there exist $\lambda, \alpha > 0$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then p_n converges to p with **order** α.

$\alpha = 1 \quad \Rightarrow \quad$ linear convergence

$\alpha = 2 \quad \Rightarrow \quad$ quadratic convergence

Order of Convergence for Fixed-Point Iteration:

- Satisfy the hypothesis of the fixed-point theorem
- If $p = g(p)$ and $g'(p) \neq 0$, then the fixed-point iteration converges **linearly**
- If $p = g(p)$, $g'(p) = 0$, and g'' is bounded in an open interval around p, then there is an interval where the fixed-point iteration converges **at least quadratically**

Example 1. Last week we showed the fixed-point iteration $x_{n+1} = g(x_n)$ with $g(x) = \arctan x + \frac{1}{2}$ converges on the interval $[1, 2]$ with some $x_0 \in [1, 2]$. What is the order of convergence?

scratch work

$g(x) = \arctan x + \frac{1}{2}$ $\qquad g(x^*) = x^* \rightarrow \arctan x^* + \frac{1}{2} = x^*$

$g'(x) = \frac{1}{1+x^2}$ $\qquad g'(x^*) \neq 0 \rightarrow$ expect linear convergence

Claim x_n converge linearly

options for proving:
- cite theorem in textbook
- directly (Mean Value Theorem)

Example 1. Last week we showed the fixed-point iteration $x_{n+1} = g(x_n)$ with $g(x) = \arctan x + \frac{1}{2}$ converges on the interval $[1,2]$ with some $x_0 \in [1,2]$. What is the order of convergence?

$g \in C[1,2]$, g is diff. on $(1,2)$ $\qquad x^* = g(x^*)$

$$|x_{n+1} - x^*| = |g(x_n) - g(x^*)|$$
$$= |g'(\xi_n)|\,|x_n - x^*|$$ by MVT where ξ_n is between x_n and x^*

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|} = |g'(\xi_n)|$$

Since ξ_n is between x_n and x^*, $\xi_n \to x^*$.

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = \lim_{n\to\infty} |g'(\xi_n)| = |g'(x^*)| = \frac{1}{1+(x^*)^2} > 0$$

Therefore $x_n \to x^*$ linearly.

Example 2. Show that the function $g(x) = \dfrac{2x^3+1}{3x^2}$ has the fixed point $x^* = 1$. Given that the fixed-point iteration $x_{n+1} = g(x_n)$ on the interval $[0.9, 1.1]$ converges to $x^* = 1$ with some $x_0 \in [0.9, 1.1]$, what is the order of convergence?

Fixed point: $g(x^*) = x^*$ i.e. $g(1) = 1$

$g(1) = \dfrac{2(1)^3+1}{3(1)^2} = \dfrac{3}{3} = 1$ ✓ $x^* = 1$ is a fixed point of g.

Scratch work: $g(x) = \frac{2}{3}x + \frac{1}{3x^2}$

$g'(x) = \frac{2}{3} - \frac{2}{3}x^{-3}$ $\qquad g'(1) = \frac{2}{3} - \frac{2}{3} = 0$

$g''(x) = 2x^{-4}$ $\qquad g''(1) = 2$ quadratic convergence

Example 2. Show that the function $g(x) = \dfrac{2x^3+1}{3x^2}$ has the fixed point $x^* = 1$. Given that the fixed-point iteration $x_{n+1} = g(x_n)$ on the interval $[0.9, 1.1]$ converges to $x^* = 1$ with some $x_0 \in [0.9, 1.1]$, what is the order of convergence?

$|x_{n+1} - x^*|$

$x^* = g(x^*)$

Claim x_n converges quadratically.

Proof. $g(x_n) = g(x^*) + \frac{g'(x^*)}{1!}(x_n - x^*) + \frac{g''(\xi_n)}{2!}(x_n - x^*)^2$ (with $g'(x^*) \to 0$)

by Taylor's Theorem where ξ_n is between x_n and x^* ($g \in C^2[0.9, 1.1]$)

$|x_{n+1} - x^*| = |g(x_n) - g(x^*)| = \frac{|g''(\xi_n)|}{2}|x_n - x^*|^2$

$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^2} = \lim_{n\to\infty} \frac{|g''(\xi_n)|}{2} = \frac{|g''(x^*)|}{2} = \frac{2}{2} = 1 > 0$

because ξ_n also converges to x^*.

Therefore $x_n \to x^*$ quadratically.

Note: it turns out the last example was Newton's method with $f(x) = x^3 - 1$.

$$x_{n+1} = g(x_n) = \frac{2x_n^3 + 1}{3x_n^2}$$

Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 1}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + 1}{3x_n^2}$$

$$= \frac{2x_n^3 + 1}{3x_n^2}$$

Order of Convergence for Newton's Method:

- $f \in C^2[a, b], p \in [a, b]$
- If $f(p) = 0$ and $f'(p) \neq 0$, then Newton's method converges **quadratically** provided we start close enough to p
- If $f(p) = 0, f'(p) = 0$, (i.e. p is not a simple zero), then Newton's method converges **linearly** provided we start close enough to p

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Example 3. Consider Newton's method of finding the root of $f(x) = 0$ where $f(x) = x^2(x-1)$ with some initial guess x_0. What are the possible roots this method could converge to? What is the order of convergence we would expect for each of these roots?

roots: $x^2(x-1)=0$

$x^*=0$ — mult. 2, not simple

$x^*=1$ — mult. 1, simple zero

- expect linear convergence near $x=0$
- expect quadratic convergence near $x=1$

$f(x) = x^3 - x^2$

$f'(x) = 3x^2 - 2x$

$f'(0) = 3(0)^2 - 2(0) = 0$ not simple

$f'(1) = 3(1)^2 - 2(1) = 1$ simple

For the last example, we run `example3.m` and `newton.m` with a tolerance of 10^{-16}.

- If we start with $x_0 = 0.1$, our method converges to $x = 0$ but it takes 24 iterations!
- If we start with $x_0 = 1.4$, our method converges to $x = 1$ in only 6 iterations.

Example 4. Consider Newton's method of finding the root of $f(x) = 0$ where $f(x) = x^2 + 2xe^{-x} + e^{-2x}$ with some initial guess $x_0 \in [0, 1]$. Does this method converge linearly or quadratically?

$f(x) = (x + e^{-x})^2$

$f(x^*) = 0$

$(x^* + e^{-x^*})^2 = 0$

$x^* + e^{-x^*} = 0$

$f'(x) = 2(x + e^{-x})(1 - e^{-x})$

$f'(x^*) = 2(x^* + e^{-x^*})(1 - e^{-x^*}) = 0$

linear convergence

$g(x^*) = x^*$

Example 5. Compare the Bisection, Fixed-Point, Newton's, Secant, and False Position methods for finding the root of $x^3 + 4x^2 - 10 = 0$ on $[1, 2]$ with a tolerance 10^{-6}.

- See `example5.m`, `bisection.m`, `fixedpoint.m`, `newton.m`, `secant.m`, `falseposition.m`
- Bisection, Newton, Secant, and False Position methods all use $f(x) = x^3 + 4x^2 - 10$
- Newton's method also requires $f'(x) = 3x^2 + 8x$
- Fixed-Point method uses $g(x) = \sqrt{\frac{10}{4+x}}$
- Initial guess for Fixed-Point and Newton methods: $p_0 = 1.5$
- Initial guesses for Secant and False Position methods: $p_0 = 1.25, p_1 = 1.5$ $(f(p_0) < 0, f(p_1) > 0)$

$x^3 + 4x^2 - 10 = 0$

$x^3 + 4x^2 = 10$

$x^2(x+4) = 10$

$x = \sqrt{\frac{10}{x+4}}$

$g_2(x) = \sqrt{\frac{10}{x+4}}$

Fixed point: $x^3 + 4x^2 - 10 = 0$

$x^3 + 4x^2 + x - 10 = x \rightarrow g_1(x) = x^3 + 4x^2 + x - 10$

$g_1'(x) = 3x^2 + 8x + 1$

diverges $|g_1'(x)| \geq 1$ on $[1,2]$

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Summary:

	Bisection	Fixed-Point	Newton's	Secant	False Position
Solves for	$f(x^*)=0$	$g(x^*)=x^*$	$f(x^*)=0$	$f(x^*)=0$	$f(x^*)=0$
Order of conv.	$\alpha=1$	$\alpha=1$ if $g'(x^*)\neq 0$ $\alpha\geq 2$ if $g'(x^*)=0$	$\alpha=1$ if $f'(x^*)=0$ $\alpha\geq 2$ if $f'(x^*)\neq 0$	$\alpha=\frac{1+\sqrt{5}}{2}$	$\alpha=\frac{1+\sqrt{5}}{2}$
Pros	• always converges • error analysis	• error analysis		no derivatives	always converges
Cons	• slow	might not converge • need a contraction	• need f' • f' close to 0	2 initial guesses	