Math 151A - Spring 2020 - Week 6

Today: Interpolation

- Power series approach
- Lagrange interpolation

Idea: Given $n+1$ points $(x_0, f_0), (x_1, f_1), ..., (x_n, f_n)$ we can find an nth degree polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ such that $P(x_i) = f_i$ for $i = 0, ..., n$.

Power Series Approach

We can build the following system of equations:

$$\begin{aligned} a_0 + a_1 x_0 + \ldots + a_n x_0^n &= f_0 \\ a_0 + a_1 x_1 + \ldots + a_n x_1^n &= f_1 \\ &\vdots \\ a_0 + a_1 x_n + \ldots + a_n x_n^n &= f_n \end{aligned}$$

This can be rewritten as the following matrix system:

$$\begin{pmatrix} 1 & x_0 & \ldots & x_0^n \\ 1 & x_1 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \ldots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Goal: solve for $\left(a_0, a_1, \ldots, a_n\right)^T$

Example 1.

(a) Write a `MATLAB` code that will find an interpolating polynomial using the power series approach.

(b) Using your `MATLAB` code, find an interpolating polynomial $P(x)$ corresponding to the data in the table below:

x_i	$f(x_i)$
0	1
1	1
3	-5

(c) Use $P(x)$ you obtained in (b) to approximate f(2).

(d) Plot $P(x)$ you obtained in (b).

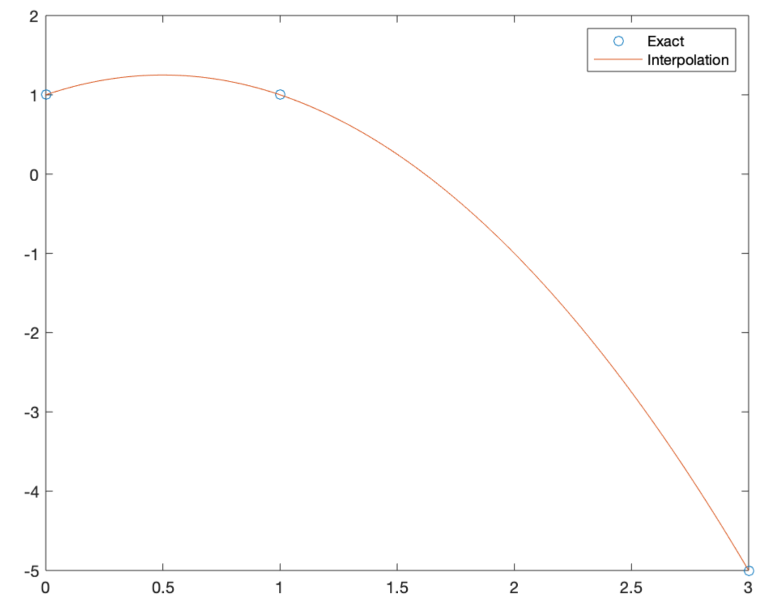
Example 1 Solutions.

(a) See `psinterp.m`.

(b) $P(x) = 1 + x - x^2$

(c) $P(2) = -1$

(d) See image to the right



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Lagrange Interpolation

The Lagrange interpolating polynomial for $(x_0, f_0), \ldots, (x_n, f_n)$ is

$$P(x) = \sum_{k=0}^{n} f_k L_{n,k}(x)$$

where $L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$.

Error analysis: If $x_0, \ldots, x_n$ are distinct numbers in $[a, b]$, $f \in C^{n+1}[a, b]$, then for each $x \in [a, b]$ there exists $\xi(x) \in (a, b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

Example 2.

(a) By hand, find the Lagrange interpolating polynomial $P(x)$ corresponding to the data in the table below:

x_i	$f(x_i)$
0	1
1	1
3	-5

(b) Use $P(x)$ you obtained in (a) to approximate f(2).

(c) Do (a) and (b) match the results in Example 1?

(d) Write a `MATLAB` code that will find the Lagrange interpolating polynomial and verify your result in (b).

Example 2. continued

x_i	f_i
0	1
1	1
3	-5

(a) Lagrange polynomial

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot f_2$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)} \cdot 1 + \frac{(x-0)(x-3)}{(1-0)(1-3)} \cdot 1 + \frac{(x-0)(x-1)}{(3-0)(3-1)} (-5)$$

$$= \boxed{\frac{1}{3}(x-1)(x-3) - \frac{1}{2}x(x-3) - \frac{5}{6}x(x-1)}$$

$\vdots$

$= -x^2 + x + 1$ (same as Example 1)

(b) $P(2) = \frac{1}{3}(2-1)(2-3) - \frac{1}{2}2(2-3) - \frac{5}{6}(2)(2-1)$

$= -\frac{1}{3} + 1 - \frac{5}{3} = 1 - 2 = -1$ (same as example 1)

(c) Yes

Example 3. Let $f(x) = \ln(x+1)$, $x_0 = 0, x_1 = 0.6, x_2 = 0.9$

(a) Construct an interpolating polynomial of degree at most two to approximate f using three points (you can use $f(0.6) = 0.47$ and $f(0.9) = 0.6$).

(b) Find an error bound for the approximation.

(c) Would your interpolating polynomial be a good estimate for $f(x)$ away from the interval $[0, 0.9]$ (e.g. at $x = 5$)? No

(a) $$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

x_i	f_i
0	0
0.6	0.47
0.9	0.6

$$P(x) = \frac{(x-0.6)\cancel{(x-0.9)}}{\cancel{(0-0.6)}(0-0.9)} \cdot \overset{0}{0} + \frac{(x-0)(x-0.9)}{(0.6-0)(0.6-0.9)} \cdot 0.47 + \frac{(x-0)(x-0.6)}{(0.9-0)(0.9-0.6)} \cdot 0.6$$

$$= \boxed{-\frac{0.47}{0.18} x(x-0.9) + \frac{0.6}{0.27} x(x-0.6)}$$

$f(x) = \ln(x+1)$

x_i	f_i
0	
0.6	
0.9	

Example 3. continued

(b) Error $f(x) - P(x) = \dfrac{f^{(n+1)}(\xi(x))}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n)$

$$|f(x) - P(x)| = \left| \frac{f^{(3)}(\xi(x))}{3!} (x-0)(x-0.6)(x-0.9) \right|$$

$$\le \max_{[0,0.9]} \left| \frac{f^{(3)}(x)}{3!} \right| |x-0||x-0.6||x-0.9|$$

$$\le \left| \frac{f^{(3)}(0)}{3!} \right| \max_{[0,0.9]} |x| \max_{[0,0.9]} |x-0.6| \max_{[0,0.9]} |x-0.9|$$

$$\le \frac{2}{3!} (0.9) |0-0.6| |0-0.9|$$

$$= \boxed{\frac{2}{6} (0.9)(0.6)(0.9)}$$

$f(x) = \ln(x+1)$

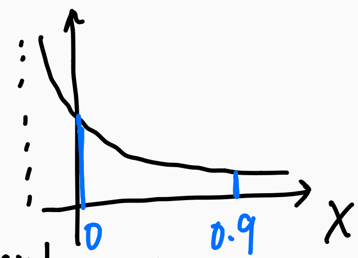
$f'(x) = (x+1)^{-1}$

$f''(x) = -(x+1)^{-2}$

$|f'''(x)| = |2(x+1)^{-3}|$

$= 2(x+1)^{-3}$

$|f^{(3)}(x)|$



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Error analysis: If $x_0, ..., x_n$ are distinct numbers in $[a, b]$, $f \in C^{n+1}[a, b]$, then for each $x \in [a, b]$ there exists $\xi(x) \in (a, b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

Proof. Define $g(t) = f(t) - P(t) - (f(x) - P(x))\dfrac{(t - x_0)(t - x_1)\cdots(t - x_n)}{(x - x_0)(x - x_1)\cdots(x - x_n)}$.
$g \in C^{n+1}[a, b]$ since $f, P \in C^{n+1}[a, b]$. Observe
$g(x) = g(x_0) = g(x_1) = ... = g(x_n) = 0$. By Generalized Rolle's Theorem there exists $\xi(x) \in [a, b]$ such that $g^{(n+1)}(\xi(x)) = 0$.

$$\begin{aligned} 0 &= g^{(n+1)}(\xi(x)) \\ &= f^{(n+1)}(\xi(x)) - P^{(n+1)}(\xi(x)) - (f(x) - P(x))\frac{(n+1)!}{(x - x_0)(x - x_1)\cdots(x - x_n)} \end{aligned}$$

But $P^{(n+1)}(\xi(x)) = 0$ since P is degree n. Solving for $f(x)$ yields

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n).$$