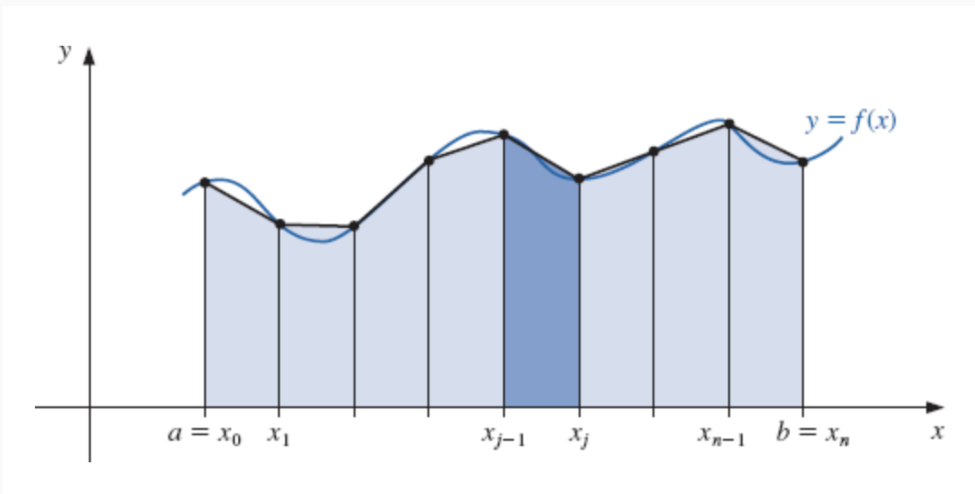
Math 151A - Spring 2020 - Week 10

Today: Numerical Integration

- Composite Rules
- Degree of Precision
- Gaussian Quadrature

Composite Trapezoidal Rule

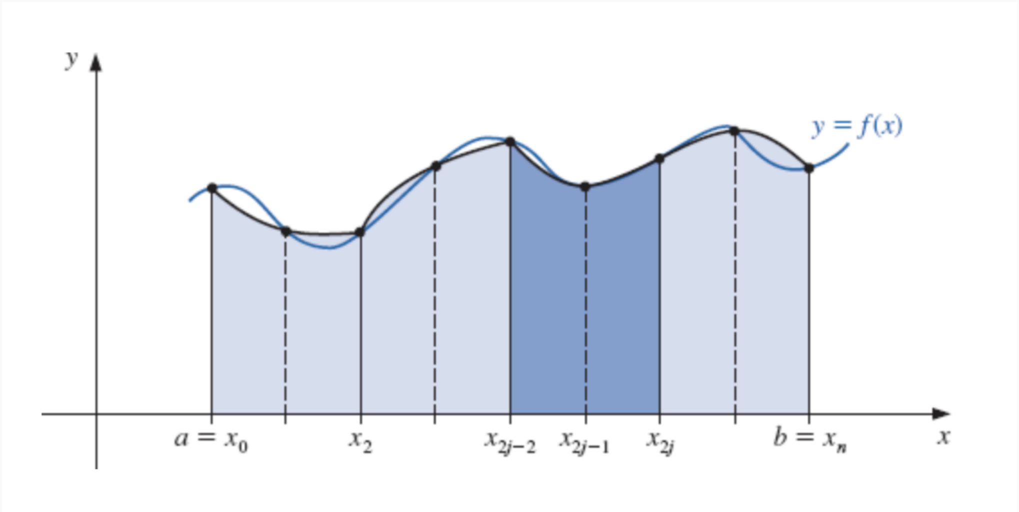
$$\int_a^b f(x)dx = h\left(\frac{1}{2}f(a) + \sum_{j=1}^{n-1} f(x_j) + \frac{1}{2}f(b)\right) \underbrace{- \frac{b-a}{12}h^2 f''(\xi)}_{\text{Error}}$$



$$h = \frac{b-a}{n} \qquad \text{mesh } x_0, \ldots, x_n$$

Composite Simpson's Rule

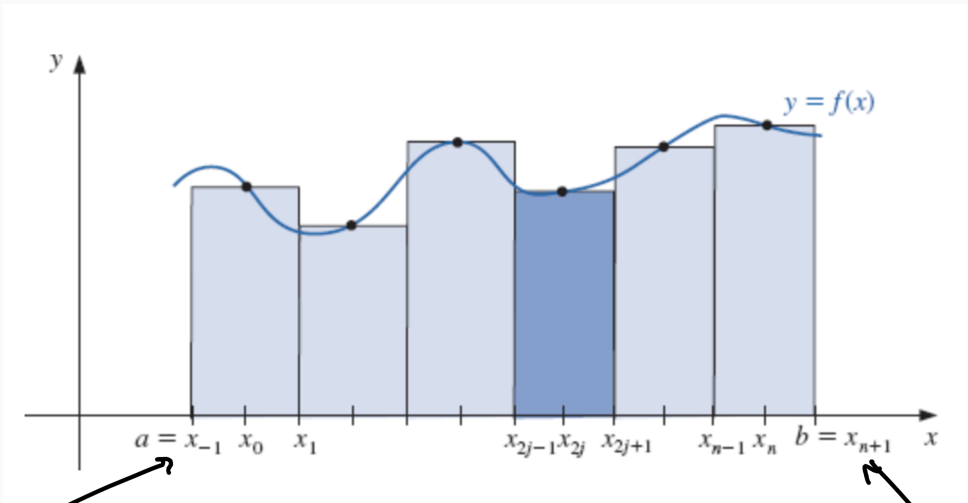
$$\int_a^b f(x)dx = \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right) \underbrace{- \frac{b-a}{180}h^4 f^{(4)}(\xi)}_{\text{Error}}$$



$h = \dfrac{b-a}{n}$ mesh $x_0, \ldots, x_n$ n even

Composite Midpoint Rule

$$\int_a^b f(x)dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \underbrace{\frac{b-a}{6}h^2 f''(\xi)}_{\text{Error}}$$



$h = \dfrac{b-a}{n+2}$ mesh $x_{-1}, x_0, \ldots, x_{n+1}$ n even

2 more points than Simpsons or trapezoidal mesh

Example 1. Determine the value of n required to approximate $\int_0^2 e^{2x} \sin 3x dx$ with an accuracy of 1×10^{-4} using the Composite Midpoint Rule.

Error: $\left| \frac{b-a}{6} h^2 f''(\xi) \right| < 1 \times 10^{-4}$

$a = 0$

$b = 2$

$h = \frac{b-a}{n+2} = \frac{2}{n+2}$

$f(x) = e^{2x} \sin 3x$

$f'(x) = 2e^{2x} \sin 3x + 3e^{2x} \cos 3x$

$f''(x) = 4e^{2x} \sin 3x + 6e^{2x} \cos 3x + 6e^{2x} \cos 3x - 9e^{2x} \sin 3x$

$= -5e^{2x} \sin 3x + 12e^{2x} \cos 3x$

$$\left| \frac{b-a}{6} h^2 f''(\xi) \right| \leq \left| \frac{b-a}{6} h^2 \right| \max_{x \in [0,2]} |f''(x)|$$

$$= \frac{2}{6} \left(\frac{2}{n+2} \right)^2 \max_{x \in [0,2]} \left| -5e^{2x} \sin 3x + 12e^{2x} \cos 3x \right|$$

Example 1. Determine the value of n required to approximate $\int_0^2 e^{2x} \sin 3x dx$ with an accuracy of 1×10^{-4} using the Composite Midpoint Rule.

$$\left| \frac{b-a}{h} h^2 f''(\xi) \right| \le \frac{2}{6}\left(\frac{2}{n+2}\right)^2 \max_{x\in[0,2]} \left(|-5e^{2x}\sin 3x| + |12e^{2x}\cos 3x| \right)$$

$$\le \frac{2}{6}\left(\frac{2}{n+2}\right)^2 \max_{x\in[0,2]} \left(|-5e^{2x}| + |12e^{2x}| \right)$$

$$\le \frac{2}{6}\left(\frac{2}{n+2}\right)^2 \left(5e^4 + 12e^4\right)$$

$$= \frac{2}{6}\left(\frac{2}{n+2}\right)^2 \left(17e^4\right)$$

$$\cancel{\le \frac{8}{n^3}(17e^4) < \frac{1}{10^4}}$$

$$\cancel{n+2 > n \quad \frac{1}{n+2} < \frac{1}{n}}$$

see next slide

$$\cancel{n^3 > 8(17e^4)(10^4) \Rightarrow n > \sqrt[3]{8(17e^4)(10^4)} \approx 420.312} \quad \boxed{n \approx 420}$$

Example 1. Determine the value of n required to approximate $\int_0^2 e^{2x} \sin 3x dx$ with an accuracy of 1×10^{-4} using the Composite Midpoint Rule.

$$\frac{2}{6}\left(\frac{2}{n+2}\right)^2 (17e^4) < \frac{1}{10^4}$$

$$\frac{8(17e^4)}{6(n+2)^2} < \frac{1}{10^4}$$

$$\frac{4}{3}(17e^4)\,10^4 < (n+2)^2$$

$$\Rightarrow \quad n+2 > \sqrt{\frac{4}{3}(17e^4)\,10^4} \approx 3517.894 \Rightarrow n+2 \approx 3518$$

$$\boxed{n \approx 3516}$$

Example 2. Write a program to approximate the integral

$$\int_0^2 x^2 e^{-x^2}\,dx$$

using the Composite Trapezoidal rule with $n = 8$, Composite Simpson's rule with $n = 8$, and the Composite Midpoint rule with $n = 6$.

see code

The **degree of precision** of a quadrature formula is the largest positive integer n such that formula is exact for x^k for each $k = 0, 1, ..., n$.

Trapezoidal rule has degree of precision 1.

Simpson's rule has degree of precision 3.

Example 3. A program was written that implements the composite trapezoidal method to approximate integrals of the form $\int_0^1 f(x)dx$. This program was tested on monomials of increasing degrees using double precision arithmetic and the number of panels $N = 1000$. The results are as follows. Is this program working correctly? Explain your answer.

Trapezoidal rule

DOP 1

$\Rightarrow$ exact for $f(x)=1$ and $f(x)=x$

$f(x)$	Computed Approximation
1	1.000500000000 $\leftarrow$ not 1
x	0.500500000000 $\leftarrow$ not $\frac{1}{2}$
x^2	0.333833500000
x^3	0.250500250000
x^4	0.200500333333

$$\int_0^1 1\,dx = x\Big|_0^1 = 1-0=1$$

$$\int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}-0=\frac{1}{2}$$

program is not working correctly because not exact for $f(x)=1$, $f(x)=x$

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Gaussian Quadrature

same functions

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

Degree of precision is $2n-1$.

x_i values come from the roots of Legendre polynomials. Some Legendre polynomials:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{3}$$

$$P_3(x) = x^3 - \frac{3}{5}x, \quad P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

Example 4.

(a) Derive the 3-point (Legendre) Gaussian Quadrature to approximate $\int_{-1}^{1} f(x)dx$.

(b) What is the degree of precision in (a)?

(c) Show that the 3-point Gaussian quadrature can be used for approximating $\int_a^b f(x)dx$ by doing a simple change of variables and applying this to approximate

$$\int_0^4 \frac{\cos x}{x+1}dx$$

(b) 3 point $n=3$

$2n-1 = 2(3)-1 = 5$

(a) $\int_{-1}^{1} f(x)\,dx = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$

Example 4. continued

(a) Derive the 3-point (Legendre) Gaussian Quadrature to approximate $\int_{-1}^{1} f(x)dx$.

$$P = x^3 - \frac{3}{5}x = 0$$
$$x\left(x^2 - \frac{3}{5}\right) = 0$$
$$x = 0 \quad x = \pm\sqrt{\frac{3}{5}}$$
$$x_1 = -\sqrt{\frac{3}{5}}, \; x_2 = 0, \; x_3 = \sqrt{\frac{3}{5}}$$

$$\int_{-1}^{1} f(x)dx = \underline{c_1} f\left(-\sqrt{\frac{3}{5}}\right) + \underline{c_2} f(0) + \underline{c_3} f\left(\sqrt{\frac{3}{5}}\right)$$

$f(x) = 1$, $f(x) = x$, $f(x) = x^2$

$f(x) = 1$ $\quad \int_{-1}^{1} 1dx = x\Big|_{-1}^{1} = 2$

$c_1 \cdot 1 + c_2 \cdot 1 + c_3 \cdot 1$

$$c_1 + c_2 + c_3 = 2$$

$f(x) = x$ $\quad \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0$

$c_1\left(-\sqrt{\frac{3}{5}}\right) + c_2 \cdot 0 + c_3\left(\sqrt{\frac{3}{5}}\right)$

$$-\sqrt{\frac{3}{5}}\,c_1 + \sqrt{\frac{3}{5}}\,c_3 = 0$$
$$-c_1 + c_3 = 0$$
$$c_1 = c_3$$

Example 4. continued

(a) Derive the 3-point (Legendre) Gaussian Quadrature to approximate $\int_{-1}^{1} f(x)dx$.

$f(x) = x^2$ $\quad \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$

$c_1\left(-\sqrt{\frac{3}{5}}\right)^2 + c_2(0)^2 + c_3\left(\sqrt{\frac{3}{5}}\right)^2$

$\frac{3}{5}c_1 + \frac{3}{5}c_3 = \frac{2}{3}$

$c_1 + c_3 = \frac{10}{9}$

$$\begin{cases} c_1 + c_2 + c_3 = 2 \\ c_1 = c_3 \\ c_1 + c_3 = \frac{10}{9} \end{cases}$$

$c_2 + \frac{10}{9} = 2$

$c_2 = \frac{8}{9}$

$2c_1 = \frac{10}{9}$

$c_1 = \frac{5}{9},\ c_3 = \frac{5}{9}$

$$\boxed{\int_{-1}^{1} f(x)dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)}$$

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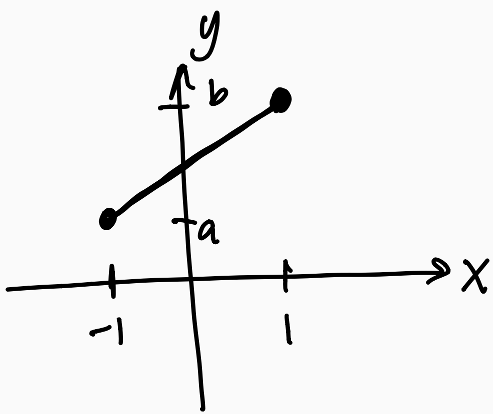
Example 4. continued $\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

(c) Show that the 3-point Gaussian quadrature can be used for approximating $\int_a^b f(x)dx$ by doing a simple change of variables and applying this to approximate

$$\int_0^4 \frac{\cos x}{x+1} dx$$

$$\int_a^b f(y)\,dy = \int_{-1}^{1} f(\underline{\ ?\ })\ ?\ dx$$

$(-1,1) \to (a,b)$



$$\text{slope} = \frac{b-a}{1-(-1)} = \frac{b-a}{2}$$

point slope $\quad y - y_1 = m(x - x_1)$

$$y - a = \frac{b-a}{2}(x+1)$$

$$y = \frac{b-a}{2}x + \frac{b-a}{2} + \frac{2a}{2}$$

$$y = \frac{(b-a)x + b+a}{2}$$

$$dy = \frac{b-a}{2} dx$$

Example 4. continued $\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

(c) Show that the 3-point Gaussian quadrature can be used for approximating $\int_a^b f(x)dx$ by doing a simple change of variables and applying this to approximate

$$\int_0^4 \frac{\cos x}{x+1} dx$$

$$\int_a^b f(y)\,dy = \int_{-1}^{1} f\left(\frac{(b-a)x + b+a}{2}\right) \frac{b-a}{2}\,dx$$

$$= \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{(b-a)x + b+a}{2}\right) dx$$

$$= \frac{b-a}{2} \left(\frac{5}{9} f\left(\frac{(b-a)\left(-\sqrt{\frac{3}{5}}\right) + b+a}{2}\right) + \frac{8}{9} f\left(\frac{b+a}{2}\right) \right.$$

$$\left. + \frac{5}{9} f\left(\frac{(b-a)\sqrt{\frac{3}{5}} + b+a}{2}\right)\right)$$

Example 4. continued

(c) Show that the 3-point Gaussian quadrature can be used for approximating $\int_a^b f(x)dx$ by doing a simple change of variables and applying this to approximate

$$\int_0^4 \frac{\cos x}{x+1} dx$$

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left(\frac{5}{9} f\left(\frac{(b-a)(-\sqrt{\frac{3}{5}})+b+a}{2}\right) + \frac{8}{9} f\left(\frac{b+a}{2}\right) + \frac{5}{9} f\left(\frac{(b-a)\sqrt{\frac{3}{5}}+b+a}{2}\right)\right)$$

$a=0 \quad b=4$

$f(x) = \frac{\cos x}{x+1}$

$$\int_0^4 \frac{\cos x}{x+1} dx \approx 2\left(\frac{5}{9} f\left(\frac{-4\sqrt{\frac{3}{5}}+4}{2}\right) + \frac{8}{9} f(2) + \frac{5}{9} f\left(\frac{4\sqrt{\frac{3}{5}}+4}{2}\right)\right)$$

$$\approx \boxed{0.279885}$$

Example 5. Show that the formula $Q(P) = \sum_{i=1}^{n} c_i P(x_i)$ cannot have degree of precision greater than $2n-1$, regardless of the choice of $c_1, \ldots, c_n$ and $x_1, \ldots, x_n$. (Hint: construct a polynomial that has double root at each of the x_i values.)

Suppose by contradiction that $Q(P)$ has degree of precision $2n$.

Consider $P(x) = (x-x_1)^2(x-x_2)^2 \cdots (x-x_n)^2$ which has degree $2n$.

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

Since DOP $2n$, then $\underbrace{\int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} c_i P(x_i)}$

on RHS $\sum_{i=1}^{n} c_i P(x_i) = 0$

on LHS $\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} \underbrace{(x-x_1)^2(x-x_2)^2\cdots(x-x_n)^2}_{\geq 0}\,dx > 0$

contradiction, $0 > 0$. Therefore DOP is at most $2n-1$. ✓