Math 151A - Spring 2020 - Week 7

Today: Interpolation

- Newton's form and divided differences
- Using `MATLAB` for interpolation and plotting
- Equally spaced nodes vs. Chebyshev nodes example
- Piecewise linear interpolation example
- Spline example
- Questions (if time)

Newton's Form

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1})$$

where

$$f[x_i] = f(x_i)$$

$$f[x_0, ..., x_k] = \frac{f[x_1, ..., x_k] - f[x_0, ..., x_{k-1}]}{x_k - x_0}$$

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Example 1. Given $(0,1),(1,2),(2,5),(3,10)$, use Newton's form and divided differences to find a polynomial $P(x)$ that fits these points. What is $P(4)$? Verify your result with `MATLAB`.

$$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$P(x) = 1 + 1(x-0) + 1(x-0)(x-1) + 0\cancel{(x-0)(x-1)(x-2)} \; 0$$

$$P(x) = 1 + x + x^2 - x$$

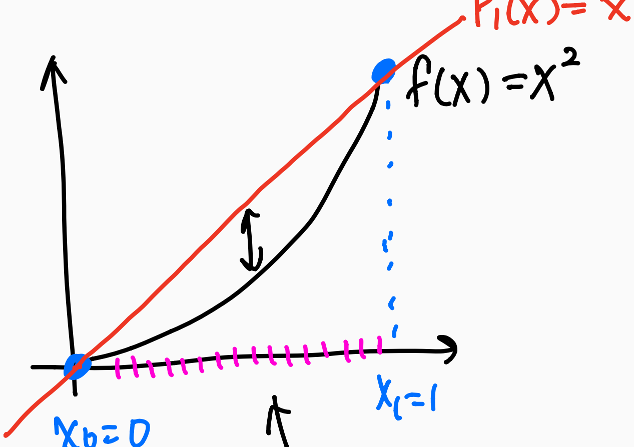
$$\boxed{P(x) = 1 + x^2}$$

$$P(4) = 1 + 4^2 = 17$$

x_i	f_i			
0	1 a_0	$\frac{2-1}{1-0} = 1$ a_1	$\frac{3-1}{2-0} = 1$ a_2	$\frac{1-1}{3-0} = 0$ a_3
1	2	$\frac{5-2}{2-1} = 3$	$\frac{5-3}{3-1} = 1$	
2	5	$\frac{10-5}{3-2} = 5$		
3	10			

Using MATLAB for interpolation and plotting

Motivating Example: Consider $f(x) = x^2$ with nodes $x_0 = 0, x_1 = 1$. What is the polynomial $P_1(x)$ that interpolates these nodes? Is $f(x_i) = P_1(x_i)$? Is $f(x) = P(x)$? Consider the error $e_1(x) = |f(x) - P_1(x)|$. What can you say about $e_1(x_i)$? Is $e_1(x) = 0$?



$$\left.\begin{array}{l} f(0) = P_1(0) \\ f(1) = P_1(1) \end{array}\right\} \rightarrow e_1(x_i) = 0$$

$$f(x) \neq P(x) \rightarrow e_1(x_i) \neq 0$$

need to use a lot more points in the interval

Using MATLAB for interpolation and plotting

Steps:

1. Generate nodes x_i, f_i for $i = 0, \ldots, n$ (options: given, equally spaced, Chebyshev)
2. Generate $\tilde{x}_j$ for plotting, $j = 0, \ldots, m$ where m is some number of plotting points (my recommendation: equally spaced)
3. Evaluate $P_n(\tilde{x}_j)$ for $j = 0, \ldots, m$
   Note: P_n depends on the nodes, so these will be an input to the MATLAB function.
   e.g. If I wanted to use Lagrange interpolation to form P_n with nodes stored in vectors $\mathbf{x}$, $\mathbf{f}$, then $P_n(x_j)$ could possibly be found with the following call:

```
P(j) = lagrange(x,f,x̃(j))
```

Example 2. Let $f(x) = \dfrac{1}{1+25x^2}$. Plot the interpolating polynomial $P_n(x)$ of $f(x)$ and error $e_n(x) = |f(x) - P_n(x)|$ on the interval $[-1, 1]$ corresponding to

(a) equally spaced nodes for $n = 4, 8, 12, 20$

(b) Chebyshev nodes for $n = 4, 8, 12, 20$.

Interval $[a, b]$

equally spaced nodes: $x_i = a + ih$, $h = \frac{b-a}{n}$, n = number of slices (maybe related to degree)

$i = 0, \cdots, n$

Chebyshev nodes: $x_i = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2i+1}{2n+1}\pi\right)$

$i = 0, \cdots, n$

Example 3. Let $f(x) = e^x$. Plot f and the piecewise linear polynomial $S_{1,5}$ that is constructed with 6 equally spaced nodes x_i using 101 equally spaced points $\tilde{x}_j$ on the interval $[0, 1]$. What is $\max_{x_i} |f(x) - S_{1,5}(x)|$? What is $\max_{\tilde{x}_j} |f(x) - S_{1,5}(x)|$?

$$\max_{x_i} |f(x) - S_{1,5}(x)| = 0$$

$$\max_{\tilde{x}_j} |f(x) - S_{1,5}(x)| \neq 0$$

0.923

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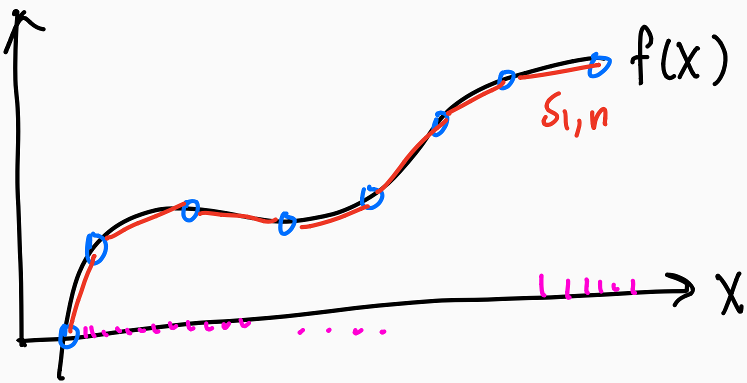
Questions?

$S_{1,n}$ n+1 points

#3a $\quad |f(x) - P_i(x)| \le \frac{1}{8} M_i h^2$ where $M_i = \max\limits_{x_{i-1}\le x\le x_i} |f''(x)|$



$x_0, \dots, x_n$ equally distributed nodes

→ $\tilde{x}_0, \dots, \tilde{x}_{999}$ equally distributed for plotting

don't affect the error

$h = \frac{b-a}{n}$

$$\text{error} \le \frac{1}{8} \max_{[0,1]} |f''(x)| \left(\frac{b-a}{n}\right)^2 < \text{tol}$$

solve for n