Math 151A - Spring 2020 - Week 8

Today: Numerical Differentiation

- Examples deriving formulas
- Applications to differential equations

Announcement: No office hour on Monday, May 25 (Memorial Day)

Example 1. Prove the following forward difference formula when f is a smooth function:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3)$$

Taylor expansion about a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Expand $f(x_0+h)$ about x_0

$$f(x_0+h) = f(x_0) + f'(x_0)(\cancel{x_0}+h-\cancel{x_0}) + \frac{f''(x_0)}{2!}(\cancel{x_0}+h-\cancel{x_0})^2$$
$$+ \frac{f'''(x_0)}{3!}(\cancel{x_0}+h-\cancel{x_0})^3 + \dots$$

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + O(h^4)$$

Example 1. continued

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(x_0) - \frac{h^2}{6} f'''(x_0) + O(h^3)$$

3.

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + O(h^4)$$

$$h f'(x_0) = f(x_0+h) - f(x_0) - \frac{h^2}{2!} f''(x_0) - \frac{h^3}{3!} f'''(x_0) + O(h^4)$$

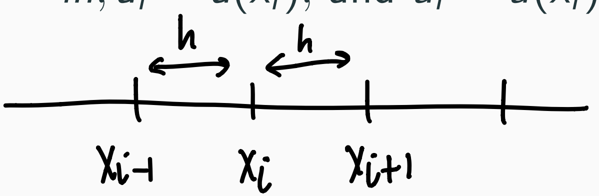
$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(x_0) - \frac{h^2}{3!} f'''(x_0) + O(h^3) \checkmark$$

order $O(h)$

Example 2. Let $u(x)$ and $a(x)$ be smooth functions. Determine the order of accuracy of

$$\frac{(a_{i+1} + a_i)(u_{i+1} - u_i) - (a_i + a_{i-1})(u_i - u_{i-1})}{2h^2}$$

as an approximation to $\left.\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)\right|_{x_i}$ where h is the mesh width, $x_i = ih$, $a_i = a(x_i)$, and $u_i = u(x_i)$.



$$x_{i+1} = x_i + h$$
$$x_{i-1} = x_i - h$$

Taylor expansions:

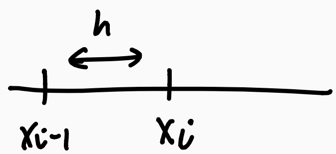
$$a(x_{i+1}) \approx a(x_i) + a'(x_i)(x_{i+1} - x_i) + \frac{a''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

$$a(x_{i+1}) = a(x_i) + h a'(x_i) + \frac{h^2}{2!} a''(x_i) + O(h^3)$$

Example 2. continued

$$\frac{(a_{i+1}+a_i)(u_{i+1}-u_i)-(a_i+a_{i-1})(u_i-u_{i-1})}{2h^2}$$

$$a(x_{i+1}) = a(x_i) + h a'(x_i) + \frac{h^2}{2!} a''(x_i) + O(h^3)$$



$$x_{i-1} = x_i - h$$

$$a_{i+1} = a_i + h a_i' + \frac{h^2}{2} a_i'' + O(h^3)$$

$$u_{i+1} = u_i + h u_i' + \frac{h^2}{2} u_i'' + O(h^3)$$

$$a(x_{i-1}) = a(x_i) + a'(x_i)(x_{i-1}-x_i) + \frac{a''(x_i)}{2!}(x_{i-1}-x_i)^2 + \cdots$$

$$= a(x_i) - h a'(x_i) + \frac{h^2}{2} a''(x_i) + O(h^3)$$

$$a_{i-1} = a_i - h a_i' + \frac{h^2}{2} a_i'' + O(h^3)$$

$$u_{i-1} = u_i - h u_i' + \frac{h^2}{2} u_i'' + O(h^3)$$

Example 2. continued

$$\frac{(a_{i+1}+a_i)(u_{i+1}-u_i)-(a_i+a_{i-1})(u_i-u_{i-1})}{2h^2}$$

$a_{i+1} = a_i + ha_i' + \frac{h^2}{2}a_i'' + O(h^3)$ $\quad$ $a_{i-1} = a_i - ha_i' + \frac{h^2}{2}a_i'' + O(h^3)$

$u_{i+1} = u_i + hu_i' + \frac{h^2}{2}u_i'' + O(h^3)$ $\quad$ $u_{i-1} = u_i - hu_i' + \frac{h^2}{2}u_i'' + O(h^3)$

Plug in:

$$\frac{1}{2h^2}\Big(\big(2a_i + ha_i' + \frac{h^2}{2}a_i'' + O(h^3)\big)\big(hu_i' + \frac{h^2}{2}u_i'' + O(h^3)\big)$$
$$-\big(2a_i - ha_i' + \frac{h^2}{2}a_i'' + O(h^3)\big)\big(hu_i' - \frac{h^2}{2}u_i'' + O(h^3)\big)\Big)$$

$$= \frac{1}{2h^2}\Big(2a_ihu_i' + h^2a_iu_i'' + h^2a_i'u_i' + O(h^3)$$
$$-\big(2ha_iu_i' - h^2a_iu_i'' - h^2a_i'u_i' + O(h^3)\big)\Big)$$

Example 2. continued

$$\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)\Big|_{x_i} = a'(x_i)u'(x_i) + a(x_i)u''(x_i)$$

$$= a_i' u_i' + a_i u_i''$$

$$\frac{(a_{i+1}+a_i)(u_{i+1}-u_i)-(a_i+a_{i-1})(u_i-u_{i-1})}{2h^2}$$

$$= \frac{1}{2h^2}\Big(\cancel{2a_i h u_i'} + h^2 a_i u_i'' + h^2 a_i' u_i' + O(h^3)$$

$$- \left(\cancel{2h a_i u_i'} - h^2 a_i u_i'' - h^2 a_i' u_i' + O(h^3)\right)\Big)$$

$$= \frac{1}{2h^2}\left(h^2 a_i u_i'' + h^2 a_i' u_i' + h^2 a_i u_i'' + h^2 a_i' u_i' + O(h^3)\right)$$

$$= \frac{1}{2h^2}\left(2h^2 a_i u_i'' + 2h^2 a_i' u_i' + O(h^3)\right)$$

$$= \underline{a_i u_i'' + a_i' u_i'} + O(h)$$

$$\boxed{\text{order } O(h)}$$

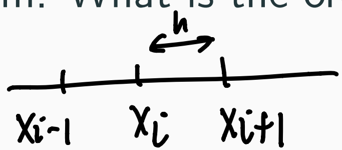
Example 2. continued

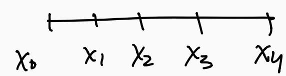
$$\frac{(a_{i+1}+a_i)(u_{i+1}-u_i)-(a_i+a_{i-1})(u_i-u_{i-1})}{2h^2}$$

Example 3. Consider the boundary-value problem

$$y'' = q(x)y + r(x), \quad \text{for } a \leq x \leq b, \quad y(a) = \alpha, y(b) = \beta$$

Using standard centered-difference approximations for $y''(x_i)$, derive the system of equations whose solution can be used to approximate the solution to this problem. What is the order of accuracy of your approximation?





$n=4$

Taylor expansions:

$$y(x_{i+1}) = y(x_i) + y'(x_i)(x_{i+1}-x_i) + \frac{y''(x_i)}{2}(x_{i+1}-x_i)^2 + \frac{y'''(x_i)}{3!}(x_{i+1}-x_i)^3 + \dots$$

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{3!} y_i''' + O(h^4)$$

Example 3. continued

$$y'' = q(x)y + r(x), \quad \text{for } a \leq x \leq b, \quad y(a) = \alpha, y(b) = \beta$$

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{3!} y_i''' + O(h^4)$$

$$+ \quad y_{i-1} = y_i - h y_i' + \frac{h^2}{2} y_i'' - \frac{h^3}{3!} y_i''' + O(h^4)$$

$$y_{i+1} + y_{i-1} = 2y_i + h^2 y_i'' + O(h^4)$$

$$\frac{h^2 y_i''}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{O(h^4)}{h^2}$$

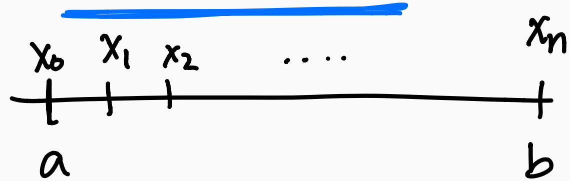
$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2)$$

$$\boxed{\text{order } O(h^2)}$$

Example 3. continued

$$y'' = q(x)y + r(x), \quad \text{for } a \le x \le b, \quad y(a) = \alpha, y(b) = \beta$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = q_i y_i + r_i$$



$y_0 = y(x_0) = y(a) = \alpha$

$y_n = y(x_n) = y(b) = \beta$

$$\frac{y_{i+1} - (q_i h^2 + 2) y_i + y_{i-1}}{h^2} = r_i$$

$i=1$: $$\frac{y_2 - (q_1 h^2 + 2) y_1 + y_0}{h^2} = r_1 \Rightarrow \frac{y_2 - (q_1 h^2 + 2) y_1}{h^2} = r_1 - \frac{\alpha}{h^2}$$

$i=2$: $$\frac{y_3 - (q_2 h^2 + 2) y_2 + y_1}{h^2} = r_2$$

$i = 2, 3, \ldots, n-2$

$i = n-1$: $$\frac{y_n - (q_{n-1} h^2 + 2) y_{n-1} + y_{n-2}}{h^2} = r_{n-1} \Rightarrow \frac{-(q_{n-1} h^2 + 2) y_{n-1} + y_{n-2}}{h^2} = r_{n-1} - \frac{\beta}{h^2}$$

Example 3. continued

$$\frac{y_{i+1} - (q_i h^2 + 2) y_i + y_{i-1}}{h^2} = r_i$$

$$y'' = q(x)y + r(x), \quad \text{for } a \le x \le b, \quad y(a) = \alpha, y(b) = \beta$$

$$\underbrace{\begin{pmatrix} -q_1 - \frac{2}{h^2} & \frac{1}{h^2} & & & \\ \frac{1}{h^2} & -q_2 - \frac{2}{h^2} & \frac{1}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{h^2} & -q_{n-2} - \frac{2}{h^2} & \frac{1}{h^2} \\ & & & \frac{1}{h^2} & -q_{n-1} - \frac{2}{h^2} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} r_1 - \frac{\alpha}{h^2} \\ r_2 \\ \vdots \\ r_{n-2} \\ r_{n-1} - \frac{\beta}{h^2} \end{pmatrix}}_{b}$$

$y = A \backslash b$

$$\frac{y_2 - (q_1 h^2 + 2) y_1}{h^2} = r_1 - \frac{\alpha}{h^2} \qquad \frac{y_3 - (q_2 h^2 + 2) y_2 + y_1}{h^2} = r_2$$

Example 3. continued

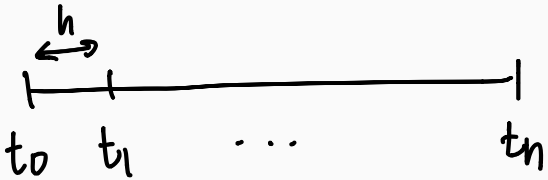
$$y'' = q(x)y + r(x), \quad \text{for } a \leq x \leq b, \quad y(a) = \alpha, y(b) = \beta$$

SIR Model: The SIR model is a simple compartmental model. The model consists of three compartments: **S**usceptible, **I**nfected, **R**ecovered. These variables represent the population in each compartment at a particular time. The SIR model can be expressed as the following system of differential equations:

$$\frac{dS}{dt} = -\beta\frac{IS}{N}$$
$$\frac{dI}{dt} = \beta\frac{IS}{N} - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

$R_0 = \dfrac{\beta}{\gamma}$ is called the "basic reproduction number", i.e. the expected number of new infections from a single infection in a population where all subjects are susceptible.

Example 4. Using the forward difference formula, program the SIR model in `MATLAB`. Try different parameters and initial values.



$$\frac{dS}{dt} = -\beta \frac{IS}{N}$$
$$\frac{dI}{dt} = \beta \frac{IS}{N} - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Forward difference

$$f_i' = \frac{f_{i+1} - f_i}{h}$$

$S(t_i) = S_i$

$I(t_i) = I_i$

$R(t_i) = R_i$

"Forward Euler Method"

$$\frac{S_{i+1} - S_i}{h} = -\beta \frac{I_i S_i}{N} \longrightarrow S_{i+1} = S_i - h\beta \frac{I_i S_i}{N}$$

$$\frac{I_{i+1} - I_i}{h} = \beta \frac{I_i S_i}{N} - \gamma I_i \longrightarrow I_{i+1} = I_i + h\left(\beta \frac{I_i S_i}{N} - \gamma I_i\right)$$

$$\frac{R_{i+1} - R_i}{h} = \gamma I_i \longrightarrow R_{i+1} = R_i + h\gamma I_i$$