Math 151A - Spring 2020 - Week 9

Today: Numerical Differentiation & Integration

Example 1. (Similar to Exercise 1 on HW) Find the error bound for the forward-difference formula

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(x_0) + O(h^2)$$

Taylor's: $\exists\ \xi$ between a and x

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(\xi)}{2!}(x-a)^2$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi)$$

ξ is between x_0 and x_0+h

$$\left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} \right| = \frac{h}{2} |f''(\xi)| \leq \max_{[x_0, x_0+h]} \frac{h}{2} |f''(\xi)|$$

central: $$\left| \frac{f(x_0+h) - f(x_0-h)}{2h} \right| \leq \max_{[x_0-h, x_0+h]?} \frac{h^2}{\ } |f'''(\xi)|$$

Example 2. (Similar to Exercise 2 on HW) Write a program that approximates the solution to the following Poisson equation in 1D for 5, 10, 15, 20 nodes:

$a=0$ $\quad \alpha = 1$

$b=1$ $\quad \beta = e = e^1$

$$-\frac{d^2y}{dx^2} = -4x^2e^{x^2} - 2e^{x^2}, \quad \boxed{y(0) = 1, y(1) = e} \quad 0 \le x \le 1$$

a $\quad b$

$\underbrace{x_0 \ x_1 \ \cdots \ x_n}_{n+1 \text{ nodes}}$

Plot against the exact solution $y = e^{x^2}$. $\longrightarrow \begin{cases} -y'' = f \\ y(a)=\alpha,\ y(b)=\beta \end{cases}$

$$\underbrace{\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}}_{\substack{n-1 \times n-1 \\ \underline{A}}} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix}}_{\underline{y}} = h^2 \underbrace{\begin{pmatrix} f_1 + \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{n-2} \\ f_{n-1} + \frac{\beta}{h^2} \end{pmatrix}}_{\underline{b}}$$

$\underline{y} = \underline{A} \backslash \underline{b}$

Example 3. (Similar to Exercise 4 on HW) Find c_1, c_2, x_1 so that the quadrature rule

$$\int_{-1}^{1} f(x)dx = c_1 f(-x_1) + c_2 f(x_1)$$

is exact for all polynomials of degree less than or equal to 3.

"works" for $f = 1, x, x^2, x^3$

$f = 1$: $\int_{-1}^{1} 1\, dx = x\Big|_{-1}^{1} = 2$

$c_1 f(-x_1) + c_2 f(x_1) = c_1 \cdot 1 + c_2 \cdot 1$

$\Rightarrow \quad c_1 + c_2 = 2$

$f = x$: $\int_{-1}^{1} x\, dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \left(\frac{1}{2}\right) = 0$

$c_1 f(-x_1) + c_2 f(x_1) = -c_1 x_1 + c_2 x_1$

$\Rightarrow \quad -c_1 x_1 + c_2 x_1 = 0$

Example 3. Continued

$$\int_{-1}^{1} f(x)dx = c_1 f(-x_1) + c_2 f(x_1)$$

$f = x^2$: $\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(\frac{-1}{3}\right) = \frac{2}{3}$

$c_1 f(-x_1) + c_2 f(x_1) = c_1 x_1^2 + c_2 x_1^2$

$c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3}$

$f = x^3$: $\int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$

$c_1 f(-x_1) + c_2 f(x_1) = -c_1 x_1^3 + c_2 x_1^3$

$-c_1 x_1^3 + c_2 x_1^3 = 0$

$$\begin{cases} c_1 + c_2 = 2 \\ -c_1 x_1 + c_2 x_1 = 0 \\ c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3} \\ -c_1 x_1^3 + c_2 x_1^3 = 0 \end{cases}$$

$-c_1 x_1 + c_2 x_1 = 0 \to (-c_1 + c_2) x_1 = 0 \to \boxed{c_1 = c_2}$ OR $\cancel{x_1 = 0}$

if $x_1 = 0$, $0 = \frac{2}{3}$ X

$c_1 + c_2 = 2$

$c_2 + c_2 = 2$

$\boxed{c_2 = 1 = c_1}$

Example 3. Continued

$$\int_{-1}^{1} f(x)dx = c_1 f\left(\underline{-x_1}\right) + c_2 f\left(\underline{x_1}\right)$$

$c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3}$

$c_1 = c_2 = 1$

$x_1^2 + x_1^2 = \frac{2}{3}$

$2x_1^2 = \frac{2}{3}$

$x_1^2 = \frac{1}{3}$

$x_1 = \pm\sqrt{\frac{1}{3}}$

$$\boxed{\int_{-1}^{1} f(x)\,dx = f\left(\sqrt{\tfrac{1}{3}}\right) + f\left(-\sqrt{\tfrac{1}{3}}\right)}$$

Special Integration Rules: (needed for Exercise 3 on HW)

Trapezoidal Rule:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b))$$

$\frac{(b-a)^3}{12} f^{(2)}(\xi)$?

Midpoint Rule:

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$$

Simpson's Rule:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$\frac{(b-a)^5}{90} f^{(4)}(\xi)$?

Example 4. The trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value of 4, and Simpson's rule gives the value of 2. What is $f(1)$?

Trapezoidal: $\frac{b-a}{2}\left(f(a)+f(b)\right)$

$4 = \frac{2-0}{2}\left(f(0)+f(2)\right) \longrightarrow 4 = f(0)+f(2)$

Simpsons: $\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$

$2 = \frac{2-0}{6}\left(f(0)+4f(1)+f(2)\right)$

$2 = \frac{1}{3}\left(f(0)+4f(1)+f(2)\right) \longrightarrow 6 = f(0)+4f(1)+f(2)$

$-4 = -f(0) \qquad -f(2)$

$2 = 4f(1)$

$\boxed{f(1) = \frac{1}{2}}$

Questions?

Taylor's:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \underbrace{\frac{f'''(a)}{3!}(x-a)^3 + \dots}$$

$\exists\ \xi_1$ between a and x s.t.

$$f(x) = f(a) + f'(a)(x-a) + \underbrace{\frac{f''(a)}{2!}(x-a)^2 + \underbrace{\frac{f'''(\xi_1)}{3!}(x-a)^3}}$$

$\exists\ \xi_2$ between a and x s.t.

$$f(x) = f(a) + f'(a)(x-a) + \underbrace{\frac{f''(\xi_2)}{2!}(x-a)^2}$$

Questions?

Forward difference

$$f(x_0+h) = f(x_0) + f'(x_0)(x_0+h-x_0) + \underbrace{\frac{f''(\xi)}{2!}(x_0+h-x_0)^2}_{(\quad O(h^2) \quad)}$$

$$\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) + \frac{f''(\xi)}{2!}h$$

Backward

$$f(x_0-h) = f(x_0) - h f'(x_0) + \frac{h^2}{2!}f''(\xi)$$