Math 1 - Summer 2020 - July 21

Important Concepts

- Distance between two points (x_1, y_1) and (x_2, y_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Equation of a circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

- Multiply degrees by $\dfrac{\pi}{180^\circ}$ to get radians
- Multiply radians by $\dfrac{180^\circ}{\pi}$ to get degrees
- To get coterminal angle add or subtract by $360^\circ(2\pi)$ until you get an angle between $0^\circ(0 \text{ rad})$ and $360^\circ(2\pi \text{ rad})$

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Important Concepts

- Arc length: $s = r\theta$ (θ in radians)
- Area of sector of circle: $A = \cancel{\tfrac{1}{2}}\theta r^2$
- If (x, y) is a point on a circle with radius r with angle θ, then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- Pythagorean theorem: $a^2 + b^2 = c^2$
- Relationships between trigonometric functions:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

- Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Example 1. Find the distance between the following pairs of points:

(a) $(5, 3)$ and $(-1, -5)$

$$d = \sqrt{(-1-5)^2 + (-5-3)^2}$$
$$= \sqrt{36+64}$$
$$= \sqrt{100}$$
$$= 10$$

(b) $(3, 3)$ and $(-3, -2)$ Try on your own

$$d = \sqrt{(-3-3)^2 + (-2-3)^2}$$
$$= \sqrt{36+25}$$
$$= \sqrt{61}$$

$(x_1, y_1), (x_2, y_2)$

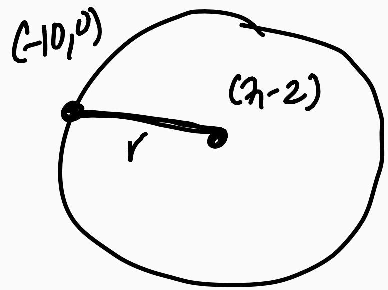
$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Example 2. Write an equation for the circle centered at $(7, -2)$ that passes through $(-10, 0)$.

circle: center (h,k), radius r

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-7)^2 + (y+2)^2 = r^2$$



$$d = \sqrt{(-10-7)^2 + (0+2)^2}$$

$$= \sqrt{17^2 + 4}$$

$$= \sqrt{289 + 4}$$

$$= \sqrt{293} = r$$

$$\boxed{(x-7)^2 + (y+2)^2 = 293}$$

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Example 3. Write an equation for the circle where the points $(-3, 3)$ and $(5, 7)$ lie along the diameter.

Try this on your own for a couple minutes

$(x-h)^2 + (y-k)^2 = r^2$

$d = \sqrt{(5+3)^2 + (7-3)^2}$

$= \sqrt{64+16}$

$2r = \sqrt{80}$

$r = \frac{\sqrt{80}}{2}$

$(x-h)^2 + (y-k)^2 = \left(\frac{\sqrt{80}}{2}\right)^2$

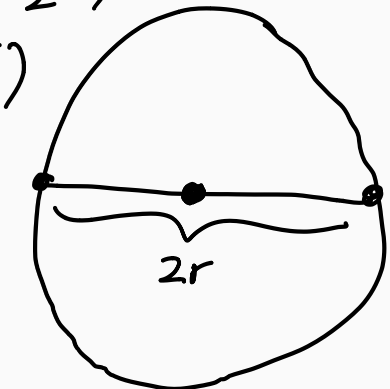
$= \frac{80}{4}$

$(x-h)^2 + (y-k)^2 = 20$

midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\left(\frac{-3+5}{2}, \frac{3+7}{2}\right)$

$(1, 5)$



$\boxed{(x-1)^2 + (y-5)^2 = 20}$

Example 4. At what point in the second [redacted] quadrant does the line with the equation $y = 2x + 5$ intersect a circle with radius 3 and center $(-2, 0)$?

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x+2)^2 + (y-0)^2 = 9$$

$$\begin{cases} (x+2)^2 + y^2 = 9 \\ y = 2x+5 \end{cases}$$

$$(x+2)^2 + (2x+5)^2 = 9$$

$$(x+2)(x+2) + (2x+5)(2x+5) = 9$$

$$x^2 + 4x + 4 + 4x^2 + 10x + 10x + 25 = 9$$

$$5x^2 + 24x + 29 = 9$$

$$5x^2 + 24x + 20 = 0$$

quadratic formula

$$x = \frac{-24 \pm \sqrt{24^2 - 4(5)(20)}}{2(5)}$$

$$= \frac{-24 \pm \sqrt{576 - 400}}{10}$$

$$= \frac{-24 \pm \sqrt{176}}{10}$$

$$\frac{-24 + \sqrt{176}}{10}$$

quadrant II

$$\frac{-24 - \sqrt{176}}{10}$$

quadrant III

Example 4. At what point in the second [redacted] quadrant does the line with the equation $y = 2x + 5$ intersect a circle with radius 3 and center $(-2, 0)$?

quadratic formula

$$x = \frac{-24 \pm \sqrt{24^2 - 4(5)(20)}}{2(5)}$$

$$= \frac{-24 \pm \sqrt{576 - 400}}{10}$$

$$= \frac{-24 \pm \sqrt{176}}{10}$$

$$x = \frac{-24 + \sqrt{176}}{10} \qquad \frac{-24 - \sqrt{176}}{10}$$

quadrant II $\qquad$ quadrant III

$y = 2x + 5$

$$y = 2\left(\frac{-24 + \sqrt{176}}{10}\right) + 5$$

$$\boxed{\left(\frac{-24 + \sqrt{176}}{10},\ 2\left(\frac{-24 + \sqrt{176}}{10}\right) + 5\right)}$$

Example 5. Convert the following angle to radians.

(a) 300°

$$\frac{300^{\circ}}{1}\cdot\frac{\pi}{180^{\circ}} = \frac{300\pi}{180} = \frac{5\pi}{3}$$

(b) 135°

Try on your own

$$135^{\circ}\cdot\frac{\pi}{180^{\circ}} = \frac{135\pi}{180} = \frac{3\pi}{4}$$

$$\boxed{\begin{array}{l}\text{Degrees} \to \text{radians}\\ \text{multiply } \dfrac{\pi}{180^{\circ}}\end{array}}$$

Example 6. Convert the following angle to degrees.

(a) $\frac{5\pi}{3}$

$$\frac{5\cancel{\pi}}{\cancel{3}} \cdot \frac{\overset{60^\circ}{\cancel{180^\circ}}}{\cancel{\pi}} = 5(60^\circ)$$

$$= 300^\circ$$

(b) $\frac{11\pi}{6}$ Try on your own

$$\frac{11\cancel{\pi}}{\cancel{6}} \cdot \frac{\overset{30^\circ}{\cancel{180^\circ}}}{\cancel{\pi}} = 11(30^\circ) = 330^\circ$$

Radians → degrees

multiply $\frac{180^\circ}{\pi}$

Example 7. Find an angle between 0° and 360° that is coterminal with an angle of

(a) 685°

$685^\circ - 360^\circ = 325^\circ$

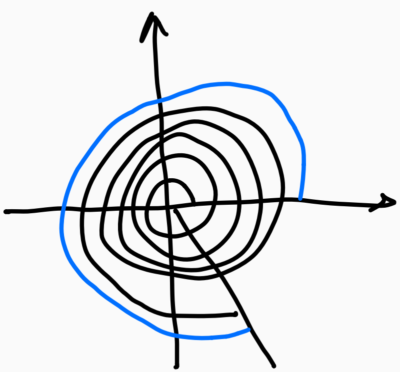
(b) -1400°

$-1400^\circ + 360^\circ = -1040^\circ$

$-1040^\circ + 360^\circ = -680^\circ$

$-680^\circ + 360^\circ = -320^\circ$

$-320^\circ + 360^\circ = 40^\circ$



Add|subtract by 360° until between 0° and 360°

Example 8. Find an angle between 0 and 2π radians that is coterminal with an angle of

(add because $\frac{-17\pi}{4} < 0$)

(a) $\frac{26\pi}{9}$

$$\frac{26\pi}{9} - \frac{2\pi}{1} = \frac{26\pi}{9} - \frac{18\pi}{9} = \frac{8\pi}{9}$$

(subtract because $\frac{26\pi}{9} > 2\pi$)

(b) $\frac{-17\pi}{4}$

Try on your own

$$\frac{-17\pi}{4} + 2\pi = \frac{-17\pi}{4} + \frac{8\pi}{4} = -\frac{9\pi}{4}$$

$$\frac{-9\pi}{4} + \frac{8\pi}{4} = -\frac{\pi}{4}$$

$$\frac{-\pi}{4} + \frac{8\pi}{4} = \frac{7\pi}{4}$$

Add / subtract by 2π until between 0 and 2π

Example 9. On a circle of radius 12 cm, find the lenth of an arc that subtends a central angle of 120°.

Arc length

$s = r\theta$

radius

angle (radians)

$r = 12\text{cm}$

$\theta = 120^\circ \cdot \frac{\pi}{180^\circ} = \frac{120\pi}{180} = \frac{2\pi}{3}$

$s = \frac{12\text{cm}}{1}\left(\frac{2\pi}{3}\right) = \frac{24\pi}{3}\text{cm} = 8\pi\ \text{cm}$

Example 10. A sector of a circle has a central angle of 30°. Find the area of the sector if the radius of the circle is 20 cm.

Area of sector

$\theta = \frac{30^\circ}{1}\left(\frac{\pi}{180^\circ}\right) = \frac{30\pi}{180} = \frac{\pi}{6}$

$r = 20\text{cm}$

$A = \theta r^2$

angle (radians)

radius

$A = \frac{\pi}{6}(20\text{cm})^2 = \frac{400\pi}{6}\text{ cm}^2$

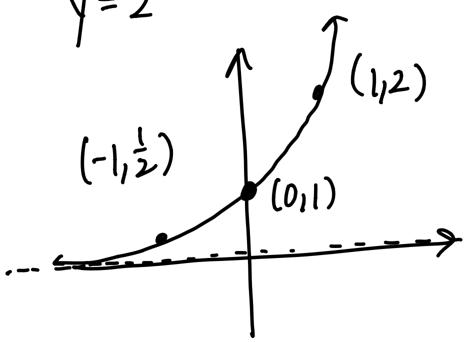
$= \boxed{\frac{200\pi}{3}\text{ cm}^2}$

Unit circle

Example 11. If $\cos\theta = \dfrac{1}{7}$ and θ is in the 4th quadrant, find $\sin\theta$.

Long run behavior of exponential functions

$y = 2^x$



$x \to -\infty$, $y = 0$ (H.A.)

$x \to \infty$, $y \to \infty$

$$y = 2^x \underbrace{+1}_{\text{H.A. } y=1}$$