UCLA Math 31B – Solutions to Practice Problems

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Handwritten solutions to practice problems for UCLA Math 31B are on the following pages. Statements are provided in a separate document on my webpage. There may be errors.

① $\int \frac{3x-1}{x^2-2x-3}\,dx$

Factor: $x^2-2x-3 = (x-3)(x+1)$

$$\frac{3x-1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$\frac{3x-1}{(x-\cancel{3})(x+\cancel{1})}(x-\cancel{3})(x+\cancel{1}) = \frac{A}{x-\cancel{3}}(x-\cancel{3})(x+1) + \frac{B}{x+\cancel{1}}(x-3)(x+\cancel{1})$$

$$3x-1 = A(x+1) + B(x-3)$$

$x=3$: $\quad 3(3)-1 = A(3+1) + \cancel{B(3-3)}^{0}$

$$8 = 4A$$

$$A=2$$

$x=-1$: $\quad 3(-1)-1 = \cancel{A(-1+1)}^{0} + B(-1-3)$

$$-4 = -4B$$

$$B=1$$

so $\int \frac{3x-1}{(x-3)(x+1)}\,dx = \int \left(\frac{2}{x-3} + \frac{1}{x+1} \right) dx$

$$= \underbrace{\int \frac{2}{x-3}\,dx}_{\text{I}} + \underbrace{\int \frac{1}{x+1}\,dx}_{\text{II}}$$

(I) $\int \frac{2}{x-3}\,dx = 2\int \frac{1}{x-3}\,dx = 2\int \frac{1}{u}\,du = 2\ln|u|$

$u = x-3$

$du = dx$

$= \underline{2\ln|x-3|}$

(II) $\int \frac{1}{x+1}\,dx = \int \frac{1}{u}\,du = \ln|u| = \underline{\ln|x+1|}$

$u = x+1$

$du = dx$

Now combine: $\boxed{2\ln|x-3| + \ln|x+1| + C}$

② Integrate $\int \frac{2x^2-x+2}{(x-2)(x^2+4)}\,dx$

partial fraction decomposition

$$\frac{2x^2-x+2}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$\frac{2x^2-x+2}{\cancel{(x-2)}\cancel{(x^2+4)}}\,(x-2)(x^2+4) = \frac{A}{\cancel{x-2}}\,\cancel{(x-2)}(x^2+4) + \frac{Bx+C}{\cancel{x^2+4}}\,(x-2)\cancel{(x^2+4)}$$

$$2x^2-x+2 = A(x^2+4) + (Bx+C)(x-2)$$

$x=2$: $2(2)^2-2+2 = A(2^2+4) + \cancel{(B\cdot 2+C)(2-2)}^{\,0}$

$8 = 8A$

$A=1$

$x=0$: $2(0)^2-0+2 = A(0^2+4) + (B\cdot 0+C)(0-2)$

$2 = 4A-2C$

$2 = 4-2C$ since $A=$

$-2 = -2C$

$C=1$

$x=1$: $2(1)^2-1+2 = A(1^2+4) + (B\cdot 1+C)(1-2)$

$3 = 5A-B-C$

$3 = 5-B-1$ since $A=1, C=1$

$3 = 4-B$

$-1=-B$

$B=1$

so $\dfrac{2x^2-x+2}{(x-2)(x^2+4)} = \dfrac{1}{x-2} + \dfrac{1x+1}{x^2+4}$

now integrate:

$$\int \frac{2x^2-x+2}{(x-2)(x^2+4)}\,dx = \int\left(\frac{1}{x-2} + \frac{x+1}{x^2+4}\right)dx$$

$$= \underbrace{\int \frac{1}{x-2}\,dx}_{\text{I}} + \underbrace{\int \frac{x}{x^2+4}\,dx}_{\text{II}} + \underbrace{\int \frac{1}{x^2+4}\,dx}_{\text{III}}$$

I $\displaystyle\int \frac{1}{x-2}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|x-2|$

$u = x-2$

$du = dx$

II $\displaystyle\int \frac{x}{x^2+4}\,dx = \int \frac{\cancel{x}}{u}\,\frac{du}{2\cancel{x}} = \frac{1}{2}\int\frac{1}{u}\,du$

$u = x^2+4$

$du = 2x\,dx \rightarrow dx = \dfrac{du}{2x}$

$= \dfrac{1}{2}\ln|u|$

$= \dfrac{1}{2}\ln|x^2+4|$

III $\displaystyle\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\left(\frac{x^2}{4}+1\right)}\,dx = \frac{1}{4}\int\frac{1}{u^2+1}\,2du$

$u^2 = \dfrac{x^2}{4}$

$= \dfrac{2}{4}\displaystyle\int\frac{1}{u^2+1}\,du$

$u = \dfrac{x}{2}$

$= \dfrac{1}{2}\arctan(u)$

$du = \dfrac{dx}{2} \rightarrow dx = 2du$

$= \dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)$

Now combine:

$$\boxed{\ln|x-2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

(3) $\int_3^6 \frac{1}{\sqrt[7]{x-5}}\,dx$

undefined at $x=5$

$$\int_3^5 \frac{1}{\sqrt[7]{x-5}}\,dx + \int_5^6 \frac{1}{\sqrt[7]{x-5}}\,dx$$

$$\lim_{a\to 5^-} \int_3^a \frac{1}{\sqrt[7]{x-5}}\,dx + \lim_{b\to 5^+} \int_b^6 \frac{1}{\sqrt[7]{x-5}}\,dx$$

Let's find the general antiderivative

$$\int \frac{1}{\sqrt[7]{x-5}}\,dx = \int \frac{1}{\sqrt[7]{u}}\,du = \int u^{-1/7}\,du$$

$u = x-5$

$du = dx$

$$= \frac{u^{6/7}}{6/7} = \frac{7}{6}u^{6/7} = \frac{7}{6}(x-5)^{6/7}$$

$$\lim_{a\to 5^-} \frac{7}{6}(x-5)^{6/7}\Big|_3^a + \lim_{b\to 5^+} \frac{7}{6}(x-5)^{6/7}\Big|_b^6$$

$$\cancel{\lim_{a\to 5^-} \frac{7}{6}(a-5)^{6/7}}^{0} - \frac{7}{6}(3-5)^{6/7}$$

$$+ \frac{7}{6}(6-5)^{6/7} - \cancel{\lim_{b\to 5^+} \frac{7}{6}(b-5)^{6/7}}^{0}$$

$$= -\frac{7}{6}(-2)^{6/7} + \frac{7}{6}(1)^{6/7}$$

$$= \boxed{-\frac{7}{6}\sqrt[7]{64} + \frac{7}{6}}$$

converges

④ $\int_1^{\infty} xe^{-x}\,dx$

$$= \lim_{a\to\infty} \int_1^{a} xe^{-x}\,dx$$

Let's find the general antiderivative

$\int xe^{-x}\,dx$

parts

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int xe^{-x}\,dx = -xe^{-x} - \int(-e^{-x})\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}$$

$$= -e^{-x}(x+1)$$

$$= -\frac{(x+1)}{e^x}$$

$$\lim_{a\to\infty} \int_1^{a} xe^{-x}\,dx = \lim_{a\to\infty} -\frac{(x+1)}{e^x}\Big|_1^a$$

$$= \underbrace{\lim_{a\to\infty} -\frac{(a+1)}{e^a}}_{\frac{-\infty}{\infty} \text{ use L'Hopital's rule}} - \left(-\frac{(1+1)}{e^1}\right)$$

$$= \lim_{a \to \infty} -\cancel{\frac{1}{e^{a}}}^{\,0} + \frac{2}{e}$$

$$= \frac{2}{e}$$

converges

(5) Arc length $L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$

$f(x) = \frac{1}{8}x^2 - \ln x$

$f'(x) = \frac{x}{4} - \frac{1}{x}$

$$L = \int_1^e \underbrace{\sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}} dx$$

simplify under this square root

$$1 + \left(\frac{x}{4}-\frac{1}{x}\right)^2 = 1 + \frac{x^2}{16} - \frac{2x}{4x} + \frac{1}{x^2}$$

$$= 1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}$$

$$= \frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}$$

common denominator $16x^2$

$$= \frac{x^4}{16x^2} + \frac{8x^2}{16x^2} + \frac{16}{16x^2}$$

$$= \frac{x^4 + 8x^2 + 16}{16x^2} \leftarrow \text{perfect square (factor)}$$

$$= \frac{(x^2+4)^2}{16x^2}$$

$$L = \int_1^e \sqrt{\frac{(x^2+4)^2}{16x^2}}\, dx = \int_1^e \frac{x^2+4}{4x}\, dx$$

$$= \int_1^e \frac{x}{4} + \frac{1}{x}\, dx = \frac{x^2}{8} + \ln x \Big|_1^e$$

$$= \frac{e^2}{8} + \ln e - \left(\frac{1^2}{8} + \cancel{\ln 1} \right)$$

$$= \frac{e^2}{8} + 1 - \frac{1}{8}$$

$$= \frac{e^2}{8} + \frac{7}{8}$$

⑥ $S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$

$f(x) = 3x+2$
$f'(x) = 3$

$$S = \int_0^4 2\pi\,(3x+2)\,\sqrt{1+(3)^2}\,dx$$

$$= 2\pi\sqrt{10}\int_0^4 (3x+2)\,dx$$

$$= 2\pi\sqrt{10}\left(\frac{3x^2}{2} + 2x\right)\Bigg|_{x=0}^{4}$$

$$= 2\pi\sqrt{10}\left(\frac{3\cdot 16}{2} + 2\cdot 4\right)$$

$$= 2\pi\sqrt{10}\left(\cancel{\frac{48}{2} + 8}^{\,32}\right)$$

$$= 64\pi\sqrt{10}$$

⑦ $f(x) = \frac{1}{x-1}$, $a=2$.

(a) $f(x) = (x-1)^{-1}$

$f'(x) = -1(x-1)^{-2}$

$f''(x) = (-1)(-2)(x-1)^{-3}$

$f'''(x) = (-1)(-2)(-3)(x-1)^{-4}$

$f^{(4)}(x) = (-1)(-2)(-3)(-4)(x-1)^{-5}$

Plug in $a=2$:

$f(2) = (2-1)^{-1} = 1$

$f'(2) = -1(2-1)^{-2} = -1$

$f''(2) = (-1)(-2)(2-1)^{-3} = (-1)(-2) = (-1)^2 2!$

$f'''(3) = (-1)(-2)(-3)(2-1)^{-4} = (-1)(-2)(-3) = (-1)^3 3!$

$f^{(4)}(3) = (-1)(-2)(-3)(-4)(2-1)^{-5} = (-1)(-2)(-3)(-4) = (-1)^4 4!$

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$$

$$= 1 - 1(x-2) + \frac{(-1)^2 \cancel{2!}}{\cancel{2!}}(x-2)^2 + \frac{(-1)^3 \cancel{3!}}{\cancel{3!}}(x-2)^3 + \frac{(-1)^4 \cancel{4!}}{\cancel{4!}}(x-2)^4$$

$$= \boxed{1 - (x-2) + (x-2)^2 - (x-2)^3 + (x-2)^4}$$

(b) In (a) we found

$$f(2) = 1$$
$$f'(2) = -1$$
$$f''(2) = (-1)^2 \, 2!$$
$$f'''(2) = (-1)^3 \, 3!$$
$$f^{(4)}(2) = (-1)^4 \, 4!$$
$$\vdots$$
$$f^{(k)}(2) = (-1)^k \, k!$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$= \sum_{k=0}^{n} \frac{(-1)^k \cancel{k!}}{\cancel{k!}} (x-2)^k$$

$$= \boxed{\sum_{k=0}^{n} (-1)^k (x-2)^k}$$

(c) Taylor remainder:

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$ where c is an unknown value between a and x

This means $$|f(x) - T_n(x)| \le \max_{a \le c \le x} \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1} \quad (*)$$

Step 1: Find an expression for $f^{(n+1)}(c)$

In part (a) we found

$$f(x) = (x-1)^{-1}$$
$$f'(x) = -1(x-1)^{-2}$$
$$f''(x) = (-1)(-2)(x-1)^{-3} = (-1)^2\, 2!\, (x-1)^{-3}$$
$$f'''(x) = (-1)(-2)(-3)(x-1)^{-4} = (-1)^3\, 3!\, (x-1)^{-4}$$
$$f^{(4)}(x) = (-1)(-2)(-3)(-4)(x-1)^{-5} = (-1)^4\, 4!\, (x-1)^{-5}$$
$$\vdots$$
$$f^{(n)}(x) = (-1)^n\, n!\, (x-1)^{-(n+1)}$$
$$f^{(n+1)}(x) = (-1)^{n+1}\,(n+1)!\,(x-1)^{-(n+2)}$$

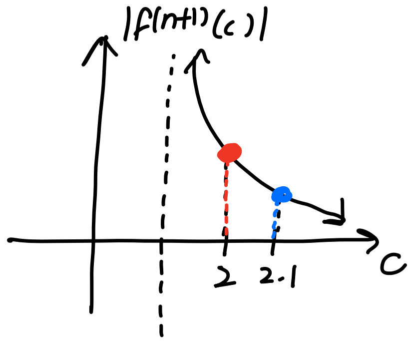
$$f^{(n+1)}(c) = (-1)^{n+1}\,(n+1)!\,(c-1)^{-(n+2)}$$

Step 2: Sketch graph of $|f^{(n+1)}(c)|$ for $a \le c \le x$

$a = 2$

$x = 2.1$ $\qquad |f^{(n+1)}(c)| = \dfrac{(n+1)!}{(c-1)^{n+2}}$

asymptote @ $c = 1$



Step 3: Find $\max\limits_{a \le c \le x} |f^{(n+1)}(c)|$ based on sketch of graph

The graph is bigger at $a=2$ vs. $x = 2.1$, so

$$\max_{a \le c \le x} |f^{(n+1)}(c)| = |f^{(n+1)}(a)| = \frac{(n+1)!}{(2-1)^{n+1}} = (n+1)!$$

Step 4: Plug max in previous step into right hand side of (*) and set less than / equal to the given value

$$\max_{a \le c \le x} \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1} \le 10^{-3}$$

$$\frac{\cancel{(n+1)}!}{\cancel{(n+1)}!} |2.1-2|^{n+1} \le 10^{-3}$$

$$(0.1)^{n+1} \le 10^{-3}$$

Step 5: Plug in values of n until the statement is true

$n=0$: $(0.1)^{0+1} \overset{?}{\le} 10^{-3}$

$0.1 \not\le 0.001$ false

$n=1$: $(0.1)^2 \overset{?}{\le} 10^{-3}$

$0.01 \not\le 0.001$ false

$n=2$: $(0.1)^3 \le 10^{-3}$

$0.001 \le 0.001$ ✓ true! $n=2$