UCLA Math 33A – Solutions to Practice Problems

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Handwritten solutions to practice problems for UCLA Math 33A are on the following pages. Statements are provided in a separate document on my webpage. There may be errors.

① Want $\vec{x} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = 0$ and $\vec{x} \cdot \begin{pmatrix} 1 \\ 0 \\ 4 \\ 3 \end{pmatrix} = 0$

$$\underbrace{2x_1 + x_2 + 0x_3 - x_4 = 0 \qquad x_1 + 0x_2 + 4x_3 + 3x_4 = 0}$$

This forms a system of equations
need to reduce

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 0 \\ 1 & 0 & 4 & 3 & 0 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 2 & 1 & 0 & -1 & 0 \end{array}\right) -2R_1 + R_2 \to R_2$$

$$\begin{array}{ccccc} -2 & 0 & -8 & -6 & 0 \\ 2 & 1 & 0 & -1 & 0 \\ \hline 0 & 1 & -8 & -7 & 0 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & -8 & -7 & 0 \end{array}\right)$$

$x_1 + 4x_3 + 3x_4 = 0$
$x_2 - 8x_3 - 7x_4 = 0$

x_3, x_4 free variables

solve for x_1 & x_2 in terms of x_3, x_4

$x_1 = -4x_3 - 3x_4$
$x_2 = +8x_3 + 7x_4$

write general solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4x_3 - 3x_4 \\ 8x_3 + 7x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -3 \\ 7 \\ 0 \\ 1 \end{pmatrix} x_4$$

$$\boxed{\vec{x} \in \text{span} \left\{ \begin{pmatrix} -4 \\ 8 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

② $V^\perp = \{\vec{x} : \vec{x} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V\}$

$= \{\vec{x} : \vec{x} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0, \ \vec{x} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0\}$

$x_1 + x_2 + 0x_3 = 0$ $\qquad$ $x_1 + 0 - x_3 = 0$

This is a system of equations

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$ $R_2 - R_1 \to R_2$ $\qquad$ need to reduce

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right)$ $-R_1 \to R_1$

$\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ \hline 0 & -1 & -1 & 0 \end{array}$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$ $R_1 - R_2 \to R_1$

$\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ \hline 1 & 0 & -1 & 0 \end{array}$

$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$ $\qquad$ $x_1 - x_3 = 0$
 $x_2 + x_3 = 0$

$\uparrow x_3$ free

Solve for x_1, x_2 in terms of x_3

$x_1 = x_3$

$x_2 = -x_3$

Write general solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} x_3$$

so $V^\perp = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$

$$\boxed{\text{basis of } V^\perp = \left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}}$$

③ (a) $\left(\begin{array}{cc|c} 1 & 1 & 6 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right) R_2 - R_1 \to R_2$

reduce

$$\begin{array}{ccc} 1 & 2 & 3 \\ -1 & -1 & -6 \\ \hline 0 & 1 & -3 \end{array}$$

$\left(\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & -3 \\ 0 & 1 & 1 \end{array}\right)$ $R_3 - R_2 \to R_3$

$$\begin{array}{ccc} 0 & 1 & 1 \\ 0 & -1 & 3 \\ \hline 0 & 0 & 4 \end{array}$$

$\left(\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{array}\right) \leftarrow 0 \neq 4$, no solution

(b) $A^T A \vec{x}^* = A^T \vec{b}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\qquad A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+1+0 & 1+2+0 \\ 1+2+0 & 1+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6+3+0 \\ 6+6+1 \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 9 \\ 3 & 6 & 13 \end{array}\right) \quad R_2 - R_1 \to R_2$$

$$\begin{array}{ccc} 3 & 6 & 13 \\ -2 & -3 & -9 \\ \hline 1 & 3 & 4 \end{array}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 9 \\ 1 & 3 & 4 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 3 & 9 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

$$\begin{array}{ccc} 2 & 3 & 9 \\ -2 & -6 & -8 \\ \hline 0 & -3 & 1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -3 & 1 \end{array}\right) \quad -\frac{1}{3} R_2 \to R_2$$

$$\left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & -1/3 \end{array}\right) \quad R_1 - 3R_2 \to R_1$$

$$\begin{array}{ccc} 1 & 3 & 4 \\ 0 & -3 & 1 \\ \hline 1 & 0 & 5 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1/3 \end{array}\right)$$

$$\boxed{\vec{x}^* = \begin{pmatrix} 5 \\ -1/3 \end{pmatrix}}$$