UCLA Math 33B – Solutions to Practice Problems

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Handwritten solutions to practice problems for UCLA Math 33B are on the following pages. Statements are provided in a separate document on my webpage. There may be errors.

① $(t^2+1)\,dy = e^{-y}\,dt$ separable

$$e^y\,dy = \frac{1}{t^2+1}\,dt$$

$$\int e^y\,dy = \int \frac{1}{t^2+1}\,dt$$

$$e^y = \arctan(t) + C$$

$y(0) = 0$ means $t=0,\ y=0$

$$e^0 = \arctan 0 + C$$

$$1 = 0 + C$$

$$C = 1$$

$$e^y = \arctan t + 1$$

$$\boxed{y = \ln(\arctan t + 1)}$$

② $y' + 3y = t^2$ first order linear

integrating factor: $\mu(t) = e^{\int 3\,dt} = e^{3t}$

$\mu'(t) = 3e^{3t}$

multiply original equation by $\mu(t)$:

$$e^{3t}(y' + 3y) = t^2 e^{3t}$$

$$e^{3t}y' + \underbrace{3e^{3t}}_{\mu'}y = t^2 e^{3t}$$

$$(e^{3t}y)' = t^2 e^{3t}$$

$$e^{3t}y = \int t^2 e^{3t}\,dt$$

parts

$u = t^2 \quad dv = e^{3t}dt$

$du = 2t\,dt \quad v = \frac{1}{3}e^{3t}$

$$= \frac{1}{3}t^2 e^{3t} - \int \frac{2}{3} t e^{3t}\,dt$$

parts again

$u = \frac{2}{3}t \quad dv = e^{3t}dt$

$du = \frac{2}{3}dt \quad v = \frac{1}{3}e^{3t}$

$$= \frac{1}{3}t^2 e^{3t} - \left(\frac{2}{9}te^{3t} - \int \frac{2}{9}e^{3t}\,dt\right)$$

$$e^{3t}y = \frac{1}{3}t^2 e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{27}e^{3t} + C$$

$$y = \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27} + Ce^{-3t}$$

$y(0)=1$ means $t=0$, $y=1$:

$$1 = \frac{1}{3}\cdot 0^2 - \frac{2}{9}\cdot 0 + \frac{2}{27} + Ce^{-0}$$

$$1 = C$$

$$\boxed{y = \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27} + e^{-3t}}$$

③ $$\underbrace{\left(2t - \frac{t}{t^2+1} + \cos(y^2)\right)}_{M} dt + \underbrace{\left(3 - 2ty\sin(y^2)\right)}_{N} dy = 0$$

exact? $M = 2t - \frac{t}{t^2+1} + \cos(y^2)$

$$\frac{\partial M}{\partial y} = -\sin(y^2) \cdot 2y = -2y\sin(y^2)$$

$N = 3 - 2ty\sin(y^2)$

$$\frac{\partial N}{\partial t} = -2y\sin(y^2)$$

same! so equation is exact

Want to find f so that $\frac{\partial f}{\partial t} = M$

$$\Rightarrow f = \int M\,dt + g(y)$$

$$= \int \left(2t - \frac{t}{t^2+1} + \cos(y^2)\right) dt + g(y)$$

$$\int 2t\,dt = t^2$$

$$\int \frac{t}{t^2+1}\,dt = \int \frac{t}{u}\,\frac{du}{2t} = \int \frac{1}{u}\,du = \ln|u| = \ln(t^2+1)$$

$u = t^2+1$, $du = 2t\,dt$

$$\int \cos(y^2)\,dt = t\cos(y^2)$$

$\Rightarrow \quad f = t^2 - \ln(t^2+1) + t\cos(y^2) + g(y)$

Want to find f so that $\frac{\partial f}{\partial y} = N$

$$\Rightarrow \frac{\partial}{\partial y}\left(t^2 - \ln(t^2+1) + t\cos(y^2) + g(y)\right) = 3 - 2ty\sin(y^2)$$

$$\cancel{-t\sin(y^2)2y} + g'(y) = 3 \cancel{-2ty\sin(y^2)}$$

$$g'(y) = 3$$

$$g(y) = \int 3\,dy$$
$$= 3y$$

$$\Rightarrow \boxed{t^2 - \ln(t^2+1) + t\cos(y^2) + 3y = C}$$

④ $2y'' + y' - 3y = 0$

$2m^2 + m - 3 = 0$

-6

$+3 \quad -2$

$2m^2 + 3m - 2m - 3 = 0$

$m(2m+3) - 1(2m+3) = 0$

$(m-1)(2m+3) = 0$

$m_1 = 1 \qquad m_2 = -3/2$

$y = c_1 e^{t} + c_2 e^{-3t/2}$

$y(0) = 3 \Rightarrow \qquad 3 = c_1 + c_2 \qquad (1)$

$y' = c_1 e^{t} + c_2 e^{-3t/2} \cdot \frac{-3}{2}$

$y'(0) = -2 \Rightarrow \qquad -2 = c_1 - \frac{3}{2} c_2 \qquad (2)$

$(1) - (2) \Rightarrow$

$$\begin{array}{rl} 3 = & c_1 + c_2 \\ +2 = & -c_1 + \frac{3}{2} c_2 \\ \hline 5 = & \frac{5}{2} c_2 \end{array}$$

$c_2 = 2$

$c_1 + 2 = 3$

$c_1 = 1$

Therefore $\boxed{y = e^{t} + 2e^{-3t/2}}$

⑤ $y'' + 6y' + 9y = 0$

$$m^2 + 6m + 9 = 0$$
$$(m+3)(m+3) = 0$$
$$(m+3)^2 = 0$$
$m = -3$ mult. 2

$$y = c_1 e^{-3t} + c_2 t e^{-3t}$$

$y(0) = 3 \Rightarrow 3 = c_1$

$$y' = c_1 e^{-3t} \cdot (-3) + c_2\left(e^{-3t} + t e^{-3t} \cdot (-3)\right)$$
$$= -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$$

$y'(0) = -10 \Rightarrow -10 = -3c_1 + c_2$

$$-10 = -9 + c_2$$
$$-1 = c_2$$

Therefore $\boxed{y = 3e^{-3t} - te^{-3t}}$

⑥ $y'' + 4y' + 13y = 0$

$m^2 + 4m + 13 = 0$ use quadratic formula

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} = \frac{-4 \pm \sqrt{16-52}}{2}$$

$$= \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -\frac{4}{2} \pm \frac{6i}{2}$$

$$= -2 \pm 3i$$

$$y = e^{-2t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

$y(0) = -1 \Rightarrow -1 = c_1$

$$y' = e^{-2t} \cdot (-2)\left(c_1 \cos(3t) + c_2 \sin(3t)\right) + e^{-2t}\left(c_1(-\sin(3t)) \cdot 3 + c_2 \cos(3t) \cdot 3\right)$$

$$= e^{-2t}\left(-2c_1 \cos(3t) - 2c_2 \sin(3t) - 3c_1 \sin(3t) + 3c_2 \cos(3t)\right)$$

$y'(0) = 14 \Rightarrow$

$$14 = -2c_1 + 3c_2$$
$$14 = 2 + 3c_2$$
$$12 = 3c_2$$
$$c_2 = 4$$

Therefore $\boxed{y = e^{-2t}\left(-\cos(3t) + 4\sin(3t)\right)}$

⑦ $y'' - y' - 2y = t + t\sin t$

homogeneous solution: $y'' - y' - 2y = 0$

$$m^2 - m - 2 = 0$$
$$(m-2)(m+1) = 0$$
$$m = 2,\ m = -1$$

$$y_h = c_1 e^{2t} + c_2 e^{-t}$$

Use undetermined coefficients to find particular solution

$g(t) = t + \sin t$ guess $y_p = At + B + C\sin t + D\cos t$

$$y_p' = A + C\cos t - D\sin t$$
$$y_p'' = -C\sin t - D\cos t$$

plug in: $y'' - y' - 2y = t + \sin t$

$$-C\sin t - D\cos t - (A + C\cos t - D\sin t) - 2(At + B + C\sin t + D\cos t) = t + \sin t$$

$$-C\sin t - D\cos t - A - C\cos t + D\sin t - 2At - 2B - 2C\sin t - 2D\cos t = t + \sin t$$

$$-2At - A - 2B + (-3C + D)\sin t + (-C - 3D)\cos t = t + \sin t$$

$$-2A = 1 \qquad -A - 2B = 0$$
$$-3C + D = 1 \qquad -C - 3D = 0$$

$-2A = 1 \Rightarrow \underbrace{A = -1/2}$

$-A-2B=0 \Rightarrow \frac{1}{2} - 2B = 0 \Rightarrow \underbrace{B = \frac{1}{4}}$

$-3C + D = 1 \Rightarrow D = 3C + 1$

$D = \frac{-9}{10} + 1$

$-C - 3D = 0 \Rightarrow -C - 3(3C+1) = 0$

$-C - 9C - 3 = 0$

$-10C = 3$

$\underbrace{C = \frac{-3}{10}}$

$\underbrace{D = \frac{1}{10}}$

Then $y_p = At + B + C\sin t + D\cos t$

$y_p = -\frac{1}{2}t + \frac{1}{4} - \frac{3}{10}\sin t + \frac{1}{10}\cos t$

$y = y_h + y_p$

$\Rightarrow \underbrace{y = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2}t + \frac{1}{4} - \frac{3}{10}\sin t + \frac{1}{10}\cos t}$

Now apply initial conditions

$y(0) = \frac{47}{20}$ means $t = 0,\ y = \frac{47}{20}$

$\frac{47}{20} = c_1 e^0 + c_2 e^0 - \frac{1}{2} \cdot 0 + \frac{1}{4} - \frac{3}{10}\sin 0 + \frac{1}{10}\cos 0$

$\frac{47}{20} = c_1 + c_2 + \frac{1}{4} + \frac{1}{10}$

$$\frac{47}{20} = c_1 + c_2 + \frac{5}{20} + \frac{2}{20}$$

$$2 = c_1 + c_2$$

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2} t + \frac{1}{4} - \frac{3}{10} \sin t + \frac{1}{10} \cos t$$

$$y' = 2c_1 e^{2t} - c_2 e^{-t} - \frac{1}{2} - \frac{3}{10} \cos t - \frac{1}{10} \sin t$$

$y'(0) = \frac{1}{5}$ means $t = 0$, $y' = \frac{1}{5}$

$$\frac{1}{5} = 2c_1 e^0 - c_2 e^0 - \frac{1}{2} - \frac{3}{10} \cos 0 - \frac{1}{10} \sin 0$$

$$\frac{1}{5} = 2c_1 - c_2 - \frac{1}{2} - \frac{3}{10}$$

$$\frac{2}{10} = 2c_1 - c_2 - \frac{5}{10} - \frac{3}{10}$$

$$1 = 2c_1 - c_2$$

$$\Rightarrow \begin{array}{r} c_1 + c_2 = 2 \\ + \; 2c_1 - c_2 = 1 \\ \hline 3c_1 = 3 \end{array}$$

$c_1 = 1$

$c_2 = 1$

$$\boxed{y = e^{2t} + e^{-t} - \frac{1}{2} t + \frac{1}{4} - \frac{3}{10} \sin t + \frac{1}{10} \cos t}$$

⑧ $y'' - 10y' + 25y = e^{5t}$

homogeneous solution: $y'' - 10y' + 25y = 0$

$$m^2 - 10m + 25 = 0$$
$$(m-5)(m-5) = 0$$
$$(m-5) = 0$$
$$m = 5 \text{ mult } 2$$

$$y_h = c_1 e^{5t} + c_2 t e^{5t}$$

Use undetermined coefficients to find particular solution

$g(t) = e^{5t}$

$\cancel{y_p = Ae^{5t}}$

This shows up in the homogeneous solution so we can't use it for our guess

$\cancel{y_p = Ate^{5t}}$

This also shows up in the homogeneous solution so we can't use it for our guess

$$y_p = At^2 e^{5t}$$

$$y_p = At^2 e^{5t}$$
$$y_p' = (At^2)' e^{5t} + At^2 (e^{5t})'$$
$$= 2At\, e^{5t} + 5At^2 e^{5t}$$
$$= (5At^2 + 2At)\, e^{5t}$$
$$y_p'' = (5At^2 + 2At)' e^{5t} + (5At^2 + 2At)(e^{5t})'$$
$$= (10At + 2A)\, e^{5t} + (5At^2 + 2At)\, 5e^{5t}$$

$$= (10At + 2A + 25At^2 + 10At)e^{5t}$$
$$= (25At^2 + 20At + 2A)e^{5t}$$

plug in: $y'' - 10y' + 25y = e^{5t}$

$$(25At^2 + 20At + 2A)\cancel{e^{5t}} - 10(5At^2 + 2At)\cancel{e^{5t}} + 25At^2\cancel{e^{5t}} = \cancel{e^{5t}}$$

$$\cancel{25At^2} + \cancel{20At} + 2A - \cancel{50At^2} - \cancel{20At} + \cancel{25At^2} = 1$$

$$2A = 1$$
$$A = \frac{1}{2}$$

$$y_p = \frac{1}{2}t^2 e^{5t}$$

$$y = y_h + y_p$$

$$y = c_1 e^{5t} + c_2 t e^{5t} + \frac{1}{2}t^2 e^{5t}$$

Now apply initial conditions

$y(0) = 2$ means $t = 0,\ y = 2$

$$2 = c_1 e^0 + c_2 \cdot 0 e^0 + \frac{1}{2} 0^2 \cdot e^0$$

$$2 = c_1$$

$$y = 2e^{5t} + c_2 t e^{5t} + \frac{1}{2}t^2 e^{5t}$$

$$y' = 10e^{5t} + (c_2 t)' e^{5t} + c_2 t (e^{5t})'$$
$$+ \left(\frac{1}{2}t^2\right)' e^{5t} + \frac{1}{2} t^2 (e^{5t})'$$

$$y' = 10e^{5t} + c_2 e^{5t} + 5c_2 t e^{5t} + te^{5t} + \frac{5}{2} t^2 e^{5t}$$

$y'(0) = 9$ means $t=0,\ y' = 9$

$$9 = 10e^0 + c_2 e^0 + 5c_2 \cdot 0 e^0 + 0e^0 + \frac{5}{2} 0^2 e^0$$

$$9 = 10 + c_2$$
$$c_2 = -1$$

$$\boxed{y = 2e^{5t} - c_2 t e^{5t} + \frac{1}{2} t^2 e^{5t}}$$

9 (a) $\vec{x}' = \underbrace{\begin{pmatrix} -5 & -7 \\ -14 & 2 \end{pmatrix}}_{A} \vec{x}$

eigenvalues: $0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & -7 \\ -14 & 2-\lambda \end{vmatrix}$

$$= (-5-\lambda)(2-\lambda) - (-14)(-7)$$
$$= -10 + 5\lambda - 2\lambda + \lambda^2 - 98$$
$$= \lambda^2 + 3\lambda - 108$$
$$= (\lambda + 12)(\lambda - 9)$$
$$\boxed{\lambda_1 = -12, \quad \lambda_2 = 9}$$

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -5-\lambda & -7 & 0 \\ -14 & 2-\lambda & 0 \end{array}\right)$

$\lambda_1 = -12$: $\left(\begin{array}{cc|c} -5-(-12) & -7 & 0 \\ -14 & 2-(-12) & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 7 & -7 & 0 \\ -14 & 14 & 0 \end{array}\right) \begin{array}{l} \frac{1}{7}R_1 \to R_1 \\ \frac{1}{14}R_2 \to R_2 \end{array}$

$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) R_1 + R_2 \to R_2 \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 = x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2 \qquad \boxed{\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\lambda_2 = 9$ $\left(\begin{array}{cc|c} -5-9 & -7 & 0 \\ -14 & 2-9 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -14 & -7 & 0 \\ -14 & -7 & 0 \end{array}\right) -R_1 + R_2 \to R_2$

$\left(\begin{array}{cc|c} -14 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right) -\frac{1}{14}R_1 \to R_1 \quad \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 = -\frac{1}{2}x_2$

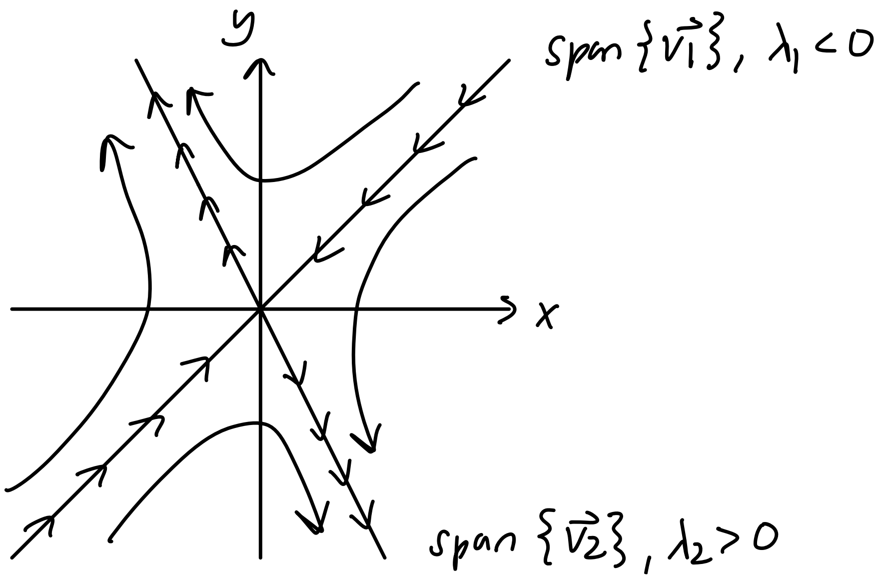
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} x_2 \qquad \boxed{\vec{v_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$

$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{9t}}$$

(b) $\lambda_1 = -12 < 0$ $\quad \lambda_2 = 9 > 0$ $\quad$ unstable

(c) saddle



(d) $\vec{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ means $t = 0$ and $\vec{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$:

$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\quad$ $2 = c_1 - c_2$
$-1 = c_1 + 2c_2$

$\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 2 & -1 \end{array}\right) \xrightarrow{-R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 3 & -3 \end{array}\right) \xrightarrow{\frac{1}{3} R_2 \to R_2}$

$\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right)$ $\quad$ $c_1 = 1$, $c_2 = -1$

$$\boxed{\vec{x} = 1\begin{pmatrix}1\\1\end{pmatrix}e^{-12t} - \begin{pmatrix}-1\\2\end{pmatrix}e^{9t}} = \begin{pmatrix} e^{-12t} + e^{9t} \\ e^{-12t} - 2e^{9t} \end{pmatrix}$$

$$\lim_{t\to\infty} \vec{x} = \begin{pmatrix}\infty\\-\infty\end{pmatrix} \quad \text{since} \quad \lim_{t\to\infty} e^{-12t} = 0, \ \lim_{t\to\infty} e^{9t} = \infty$$

(10) (a) $\vec{x}' = \underbrace{\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}}_{A} \vec{x}$

eigenvalues: $0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 0 \\ 0 & -5-\lambda \end{vmatrix}$

$= (-5-\lambda)(-5-\lambda) - 0 \cdot 0$

$= (\lambda+5)^2$

$\boxed{\lambda = -5 \quad \text{mult. } 2}$

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0}$ $\quad \left(\begin{array}{cc|c} -5-\lambda & 0 & 0 \\ 0 & -5-\lambda & 0 \end{array}\right)$

$\lambda = -5$ $\quad \left(\begin{array}{cc|c} -5-(-5) & 0 & 0 \\ 0 & -5-(-5) & 0 \end{array}\right)$ $\quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

x_1, x_2 free

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2$

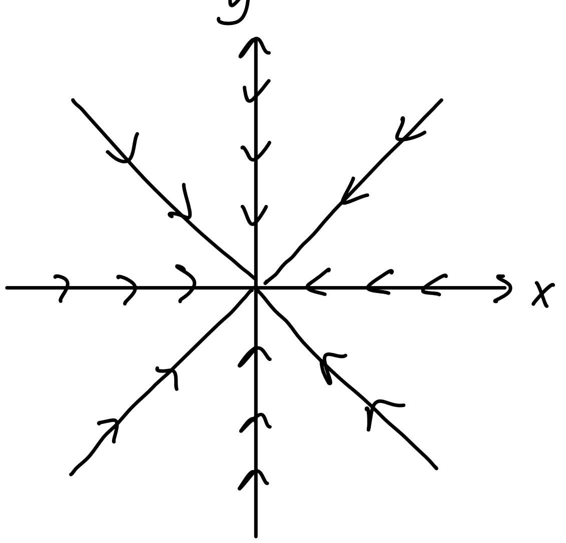
$\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$

$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t}}$

(b) $\lambda = -5 < 0$ stable

(c) nodal sink



(d) $\vec{x}(0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ means $t=0$ and $\vec{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{aligned} c_1 &= -2 \\ c_2 &= 3 \end{aligned}$$

$$\boxed{\vec{x} = -2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-5t} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t}} = \begin{pmatrix} -2e^{-5t} \\ 3e^{-5t} \end{pmatrix}$$

$\lim_{t\to 0} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since $\lim_{t\to\infty} e^{-5t} = 0$.

(11) (a) $\vec{x}' = \underbrace{\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}}_{A} \vec{x}$

eigenvalues: $0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 0 \cdot 1$

$= (2-\lambda)^2$

$\boxed{\lambda = 2 \quad \text{mult } 2}$

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} 2-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \end{array}\right)$

$\lambda = 2$: $\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_2 = 0$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1$ $\qquad \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

We only got one eigenvector but the eigenvalue has multiplicity 2, so we need to find the generalized eigenvector

$(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$\left(\begin{array}{cc|c} 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{cc|c} 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_2 = -1$

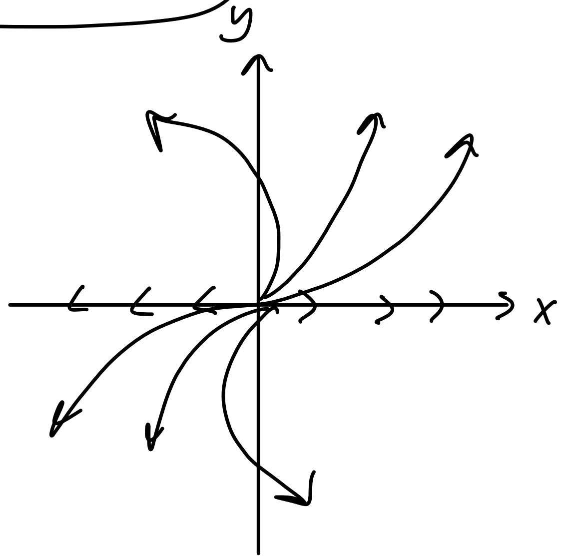
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{v}_1} x_1 + \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\vec{v}_2}$ $\qquad \boxed{\vec{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}}$

$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 e^{\lambda_1 t}(\vec{v}_1 t + \vec{v}_2)$

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)}$$

(b) $\lambda = 2 > 0$ Unstable

(c) degenerate nodal source



span$\{\vec{v}_1\}$, $\lambda > 0$

(d) $\vec{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ mean $t=0$ and $\vec{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \begin{matrix} c_1 = 3 \\ c_2 = -5 \end{matrix}$$

$$\boxed{\vec{x} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} - 5e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)} = \begin{pmatrix} 3e^{2t} - 5te^{2t} \\ 5e^{2t} \end{pmatrix}$$

$\lim_{t \to \infty} \vec{x}(t) = \begin{pmatrix} -\infty \\ \infty \end{pmatrix}$ since

$$\lim_{t \to \infty} 3e^{2t} - 5te^{2t} = \lim_{t \to \infty} e^{2t}(3 - 5t) = -\infty$$

and $\lim_{t \to \infty} e^{2t} = \infty$.

(12) (a) $\vec{x}' = \underbrace{\begin{pmatrix} -5 & 10 \\ -2 & 3 \end{pmatrix}}_{A} \vec{x}$

eigenvalues: $0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 10 \\ -2 & 3-\lambda \end{vmatrix}$

$= (-5-\lambda)(3-\lambda) - (-2)10$

$= -15 + 5\lambda - 3\lambda + \lambda^2 + 20$

$= \lambda^2 + 2\lambda + 5$

quadratic formula $\lambda = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \dfrac{-2 \pm \sqrt{-16}}{2}$

$= \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

$\boxed{\lambda_1 = -1 + 2i, \quad \lambda_2 = -1 - 2i}$

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -5-\lambda & 10 & 0 \\ -2 & 3-\lambda & 0 \end{array}\right)$

$\lambda = -1 + 2i$: $\left(\begin{array}{cc|c} -5-(-1+2i) & 10 & 0 \\ -2 & 3-(-1+2i) & 0 \end{array}\right)$

$\left(\begin{array}{cc|c} -4-2i & 10 & 0 \\ -2 & 4-2i & 0 \end{array}\right)$ $\begin{array}{l} (-4+2i)R_1 \to R_1 \\ -\frac{1}{2}R_2 \to R_2 \end{array}$

$(-4-2i)(-4+2i) = 16 - \cancel{8i} + \cancel{8i} - 4i^2 = 16 + 4 = 20$

$10(-4+2i) = -40 + 20i$

$\left(\begin{array}{cc|c} 20 & -40+20i & 0 \\ 1 & -2+i & 0 \end{array}\right)$ $R_1 \leftrightarrow R_2$

$\left(\begin{array}{cc|c} 1 & -2+i & 0 \\ 20 & -40+20i & 0 \end{array}\right)$ $-20R_1 + R_2 \to R_2$

$\left(\begin{array}{cc|c} 1 & -2+i & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{array}{l} x_1 + (-2+i)x_2 = 0 \\ x_1 = (2-i)x_2 \end{array}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (2-i)x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} x_2 \qquad \boxed{\vec{v}_1 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}}$$

$$\boxed{\vec{v}_2 = \overline{\vec{v}_1} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}}$$

$$\vec{x} = c_1 \, \mathrm{Re}(\vec{v}_1 e^{\lambda_1 t}) + c_2 \, \mathrm{Im}(\vec{v}_1 e^{\lambda_1 t})$$

$$\vec{v}_1 e^{\lambda_1 t} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{(-1+2i)t} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{-t} \underbrace{e^{2it}}_{\text{Euler's identity}}$$

$$e^{2it} = \cos 2t + i\sin 2t$$

$$= \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{-t} (\cos 2t + i\sin 2t)$$

$$= \begin{pmatrix} (2-i)(\cos 2t + i\sin 2t) \\ 1\,(\cos 2t + i\sin 2t) \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} 2\cos 2t + 2i\sin 2t - i\cos 2t - i^2 \sin 2t \\ \cos 2t + i\sin 2t \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} 2\cos 2t + 2i\sin 2t - i\cos 2t + \sin 2t \\ \cos 2t + i\sin 2t \end{pmatrix} e^{-t}$$

$$= \left[\begin{pmatrix} 2\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} 2\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \right] e^{-t}$$

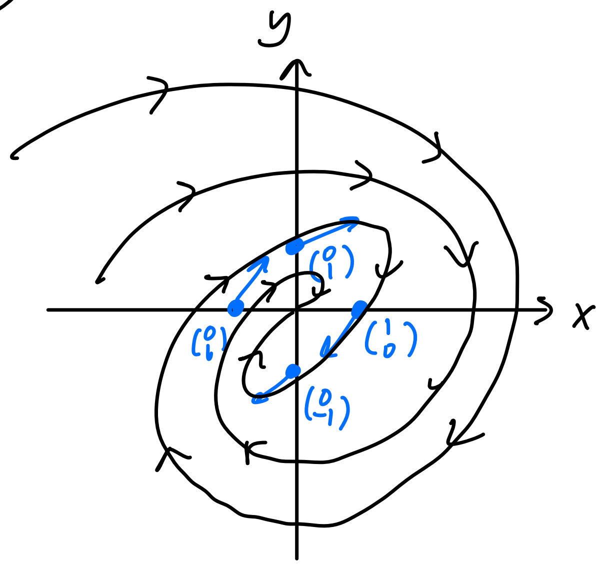
$$\mathrm{Re}(\vec{v}_1 e^{\lambda_1 t}) = \begin{pmatrix} 2\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} e^{-t}$$

$$\mathrm{Im}(\vec{v}_1 e^{\lambda_1 t}) = \begin{pmatrix} 2\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} e^{-t}$$

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 2\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} e^{-t}}$$

(b) $\lambda = -1 \pm i$ $\quad$ $Re(\lambda) < 0$ $\quad$ stable

(c) stable spiral



Check direction, plug in some points

$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\quad \begin{pmatrix} -5 & 10 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$ left down

$\vec{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$: $\quad \begin{pmatrix} -5 & 10 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ right up

$\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $\quad \begin{pmatrix} -5 & 10 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$ right up

$\vec{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$: $\quad \begin{pmatrix} -5 & 10 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ -3 \end{pmatrix}$ left down

(d) $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ means $t = 0$ and $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \begin{aligned} 2c_1 - c_2 &= 1 \\ c_1 &= 1 \end{aligned}$$

$$\begin{aligned} 2 - c_2 &= 1 \\ c_2 &= 2 - 1 \\ c_2 &= 1 \end{aligned}$$

$$\boxed{\vec{x} = \begin{pmatrix} 2\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} e^{-t} + \begin{pmatrix} 2\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} e^{-t}}$$

$$= \begin{pmatrix} 2\cos 2t + \sin 2t + 2\sin 2t - \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} \cos 2t + 3\sin 2t \\ \cos 2t + \sin 2t \end{pmatrix} e^{-t}$$

$\lim_{t \to \infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since $\lim_{t \to \infty} e^{-t} = 0$.

(13) $\vec{x}' = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{-t} \\ t \end{pmatrix}$

homogeneous solution: $\vec{x}' = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix} \vec{x}$

eigenvalues: $\begin{vmatrix} 4-\lambda & 1 \\ 2 & 5-\lambda \end{vmatrix} = (4-\lambda)(5-\lambda) - 1\cdot 2$

$$= 20 - 9\lambda + \lambda^2 - 2$$
$$= \lambda^2 - 9\lambda + 18$$
$$= (\lambda-3)(\lambda-6) = 0$$
$$\underline{\lambda_1 = 3 \quad \lambda_2 = 6}$$

eigenvectors: $\left(\begin{array}{cc|c} 4-\lambda & 1 & 0 \\ 2 & 5-\lambda & 0 \end{array}\right)$

$\lambda_1 = 3$: $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right) \; -2R_1 + R_2 \to R_2 \quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 + x_2 = 0$, $\; x_1 = -x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2 \qquad \underline{\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

$\lambda_2 = 6$: $\left(\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right) \; R_1 + R_2 \to R_2 \quad \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \; -\frac{1}{2}R_1 \to R_1$

$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 - \frac{1}{2}x_2 = 0$, $\; x_1 = \frac{1}{2}x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} x_2 \qquad \underline{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$$\vec{x}_h = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t}$$

particular solution

<u>one way</u> undetermined coefficients

$$\vec{g}(t) = \begin{pmatrix} e^{-t} \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$$

Guess $\vec{x_p} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$\vec{x_p}' = -\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

plug in:

$$-\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix} \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right] + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$$

$$\begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix} e^{-t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_2 \\ 2a_1 + 5a_2 \end{pmatrix} e^{-t} + \begin{pmatrix} 4b_1 + b_2 \\ 2b_1 + 5b_2 \end{pmatrix} t + \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 5c_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$$

$$-a_1 = 4a_1 + a_2 + 1 \qquad 0 = 4b_1 + b_2 \qquad b_1 = 4c_1 + c_2$$
$$-a_2 = 2a_1 + 5a_2 \qquad 0 = 2b_1 + 5b_2 + 1 \qquad b_2 = 2c_1 + 5c_2$$

$$5a_1 + a_2 = -1 \qquad 4b_1 + b_2 = 0 \qquad 4c_1 + c_2 = b_1$$
$$2a_1 + 6a_2 = 0 \qquad 2b_1 + 5b_2 = -1 \qquad 2c_1 + 5c_2 = b_2$$

$\times -6$ $\times -2$

$$-30a_1 - 6a_2 = 6 \qquad 4b_1 + b_2 = 0$$
$$+\ 2a_1 + 6a_2 = 0 \qquad +\ -4b_1 - 10b_2 = 2$$

$$-28a_1 = 6$$
$$a_1 = -\frac{3}{14}$$

$$2a_1 + 6a_2 = 0$$
$$a_2 = -\frac{1}{3}a_1 = \frac{1}{14}$$

$$-9b_2 = 2$$
$$b_2 = -2/9$$
$$4b_1 + b_2 = 0$$
$$b_1 = -\frac{1}{4}b_2 = -\frac{1}{4}\cdot\frac{-2}{9} = \frac{1}{18}$$

$$4c_1 + c_2 = b_1 \Rightarrow 4c_1 + c_2 = \frac{1}{18} \Rightarrow 4c_1 + c_2 = \frac{1}{18}$$
$$2c_1 + 5c_2 = b_2 \qquad 2c_1 + 5c_2 = -\frac{2}{9} \xrightarrow{\times -2} -4c_1 - 10c_2 = \frac{4}{9}$$
$$-9c_2 = \frac{9}{18}$$
$$c_2 = -\frac{1}{18}$$

$$4c_1 = \frac{1}{18} - c_2$$
$$4c_1 = \frac{1}{9}$$
$$c_1 = \frac{1}{36}$$

so $$\vec{x_p} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= \begin{pmatrix} -3/14 \\ 1/14 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/18 \\ -2/9 \end{pmatrix} t + \begin{pmatrix} 1/36 \\ -1/18 \end{pmatrix}$$

$$\vec{x} = \vec{x_h} + \vec{x_p}$$

$$\boxed{\vec{x} = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t} + \begin{pmatrix} -3/14 \\ 1/14 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/18 \\ -2/9 \end{pmatrix} t + \begin{pmatrix} 1/36 \\ -1/18 \end{pmatrix}}$$

another way use the fundamental matrix

$$\vec{x}_h = c_1\begin{pmatrix}-1\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e^{6t}$$

$$\Phi = \begin{pmatrix}-e^{3t} & e^{6t}\\ e^{3t} & 2e^{6t}\end{pmatrix}$$

$$\Phi^{-1} = \frac{1}{-2e^{9t}-e^{9t}}\begin{pmatrix}2e^{6t} & -e^{6t}\\ -e^{3t} & -e^{3t}\end{pmatrix} = \frac{1}{-3e^{9t}}\begin{pmatrix}2e^{6t} & -e^{6t}\\ -e^{3t} & -e^{3t}\end{pmatrix}$$

$$= \frac{-1}{3}\begin{pmatrix}2e^{-3t} & -e^{-3t}\\ -e^{-6t} & -e^{-6t}\end{pmatrix}$$

$$\vec{u}' = \Phi^{-1}\vec{g}(t) = \frac{-1}{3}\begin{pmatrix}2e^{-3t} & -e^{-3t}\\ -e^{-6t} & -e^{-6t}\end{pmatrix}\begin{pmatrix}e^{-t}\\ t\end{pmatrix}$$

$$= \frac{-1}{3}\begin{pmatrix}2e^{-4t} - te^{-3t}\\ -e^{-7t} - te^{-6t}\end{pmatrix}$$

$$\vec{u} = \frac{-1}{3}\int\begin{pmatrix}2e^{-4t} - te^{-3t}\\ -e^{-7t} - te^{-6t}\end{pmatrix}dt$$

$$u_1 = \frac{-1}{3}\int\left(2e^{-4t} - te^{-3t}\right)dt = \frac{-2}{3}\int e^{-4t}\,dt + \frac{1}{3}\int te^{-3t}\,dt$$

parts
$u = t \qquad dv = e^{-3t}dt$
$du = dt \qquad v = -\frac{1}{3}e^{-3t}$

$$= +\frac{2}{12}e^{-4t} + \frac{1}{3}\left(-\frac{1}{3}te^{-3t} + \int +\frac{1}{3}e^{-3t}\,dt\right)$$

$$= \frac{1}{6} e^{-4t} - \frac{1}{9} t e^{-3t} - \frac{1}{27} e^{-3t}$$

$$u_2 = -\frac{1}{3}\int (e^{-7t} - t e^{-6t})\,dt = -\frac{1}{3}\int e^{-7t}\,dt + \frac{1}{3}\int t e^{-6t}\,dt$$

parts $\quad u = t \quad dv = e^{-6t}\,dt$

$\quad du = dt \quad v = -\frac{1}{6} e^{-6t}$

$$= -\frac{1}{21} e^{-7t} + \frac{1}{3}\left(-\frac{1}{6} t e^{-6t} + \int \frac{+1}{6} e^{-6t}\,dt\right)$$

$$= -\frac{1}{21} e^{-7t} - \frac{1}{18} t e^{-6t} - \frac{1}{108} e^{-6t}$$

$$\vec{u} = \begin{pmatrix} \frac{1}{6} e^{-4t} - \frac{1}{9} t e^{-3t} - \frac{1}{27} e^{-3t} \\ -\frac{1}{21} e^{-7t} - \frac{1}{18} t e^{-6t} - \frac{1}{108} e^{-6t} \end{pmatrix}$$

$$\vec{x}_p = \Phi \vec{u} = \begin{pmatrix} -e^{3t} & e^{6t} \\ e^{3t} & 2e^{6t} \end{pmatrix} \begin{pmatrix} \frac{1}{6} e^{-4t} - \frac{1}{9} t e^{-3t} - \frac{1}{27} e^{-3t} \\ -\frac{1}{21} e^{-7t} - \frac{1}{18} t e^{-6t} - \frac{1}{108} e^{-6t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{6} e^{-t} + \frac{1}{9} t + \frac{1}{27} - \frac{1}{21} e^{-t} - \frac{1}{18} t - \frac{1}{108} \\ \frac{1}{6} e^{-t} - \frac{1}{9} t - \frac{1}{27} - \frac{2}{21} e^{-t} - \frac{1}{9} t - \frac{1}{54} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{14} e^{-t} + \frac{1}{18} t + \frac{1}{36} \\ \frac{1}{14} e^{-t} - \frac{2}{9} t - \frac{1}{18} \end{pmatrix}$$

$$= \begin{pmatrix} -3/14 \\ 1/14 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/18 \\ -2/9 \end{pmatrix} t + \begin{pmatrix} 1/36 \\ -1/18 \end{pmatrix}$$

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

$$\boxed{\vec{x} = c_1\begin{pmatrix}-1\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e^{6t} + \begin{pmatrix}-3/14\\1/14\end{pmatrix}e^{-t} + \begin{pmatrix}1/18\\-2/9\end{pmatrix}t + \begin{pmatrix}1/36\\-1/18\end{pmatrix}}$$